20/7/2015

ECI 2015
Buenos Aires

Fundamentos lógicos de bases de datos
(Logical foundations of databases)

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Recap

- **Relational model** (tables)
- **Relational Algebra** (union, product, difference, selection, projection)
- **SQL** (SELECT … FROM … WHERE …)
- **First-order logic** (syntax, semantics, active domain)
- **Expressiveness** (FO^{act} = RA = basic SQL)
- **Undecidable problems** (Halting ≤ Domino ≤ Satisfiability ≤ Equivalence)
- **Data complexity / Combined complexity**
- **Complexity of evaluation** (LOGSPACE / PSPACE complexity)

Definability in FO

Goal: check which properties / queries are *expressible in FO*

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Example. $Q(G) = \{ (u, v) \mid G \text{ contains a path from } u \text{ to } v \}$

Is Q expressible as a first-order formula?

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Quantifier rank ≠ quantity of quantifiers

Eg, in $d_0(x, y) = E(x, y)$, and

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Sub-goal: Given a property P and a number n,
tell whether P is expressible by a sentence of quantifier rank at most n.

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Definition. Two structures S_1 and S_2 are ***n*-equivalent** [Tarski '30]

iff

they satisfy the same FO sentences of quantifier rank ≤ n

(i.e. $S_1 \vDash \phi$ iff $S_2 \vDash \phi$ for all $\phi \in$ FO with $qr(\phi) \leq n$)

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Suppose that there are $S_1 \in \boldsymbol{P}$, $S_2 \notin \boldsymbol{P}$ *s.t.*

S_1 and S_2 are *n - equivalent*.

Then $\boldsymbol{P}$ is *not expressible* by any sentence of quantifier rank n.

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Example. **P** = { finite structures } seems to be not FO-definable.
One could then aim at proving that

for all n there are $S_1 \in P$ and $S_2 \notin P$ s.t. S_1, S_2 n-equivalent...

Expressive power via games

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Characterization of the expressive power of FO in terms of **Games**

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Idea: For every two structures (S,S') there is a game where

a player of the game has a **winning strategy**

iff

S,S' are **indistinguishable**

Ehrenfeucht-Fraïssé games

A game between two players

Board: (S_1, S_2)

One player plays in one structure, the other player answers in the other structure.

If after n rounds Duplicator doesn't lose: S_1, S_2 are n-equivalent

Ehrenfeucht-Fraïssé games

Definition. **Partial isomorphism** between S_1 and S_2 = injective partial map

$$f\colon \text{nodes of } S_1 \longrightarrow \text{nodes of } S_2$$

so that $E(x,y)$ iff $E(f(x),f(y))$

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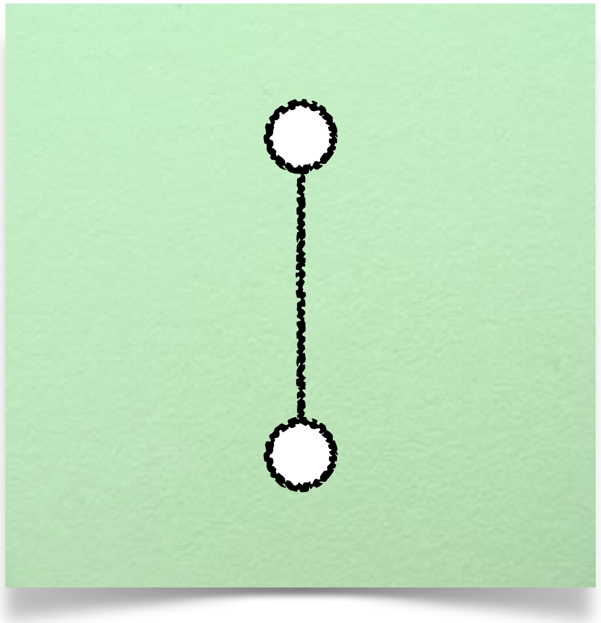
   and **Duplicator** answers with a node x_i from S_1,

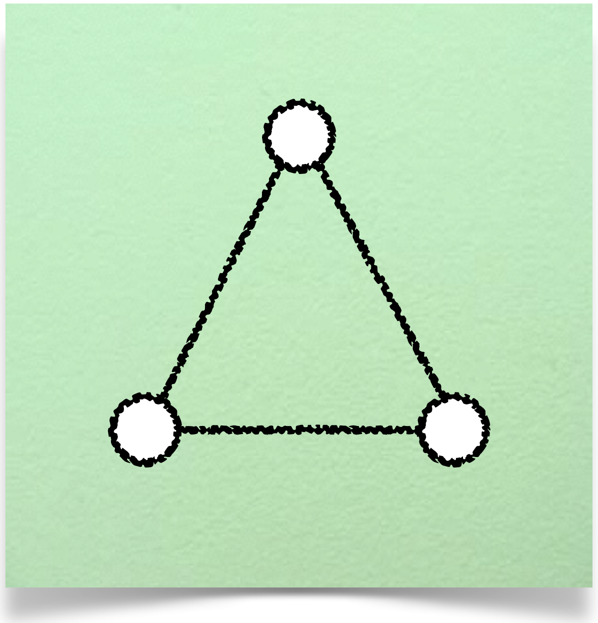
or Spoiler wins if $\{ x_i \mapsto y_i \mid 1 \leq i \leq n \}$ is **not a partial isomorphism** between S_1 and S_2.

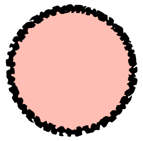
Ehrenfeucht-Fraïssé games

S_1

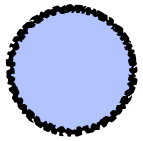


S_2





= Spoiler



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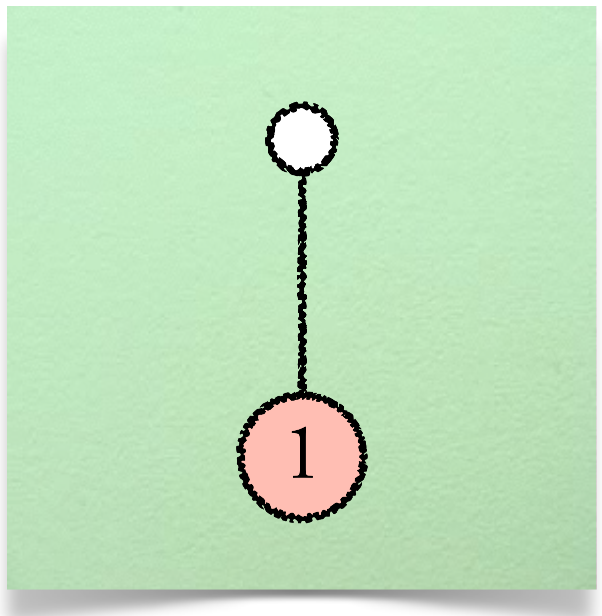

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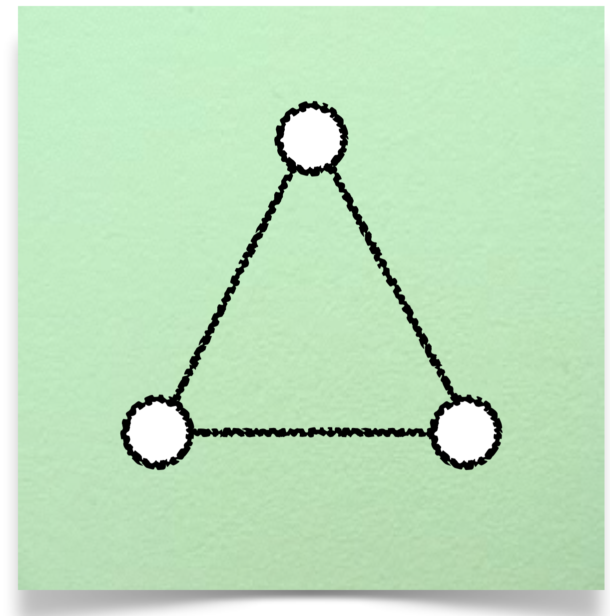
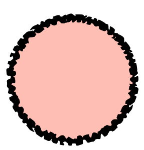
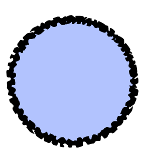


 = Spoiler

 = Duplicator

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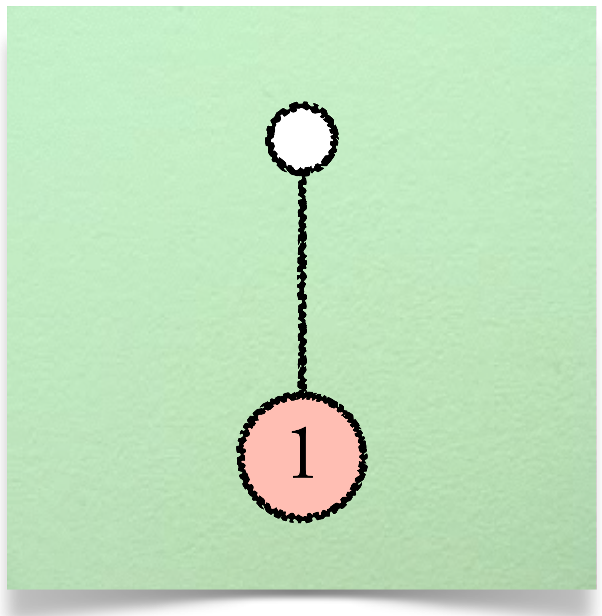

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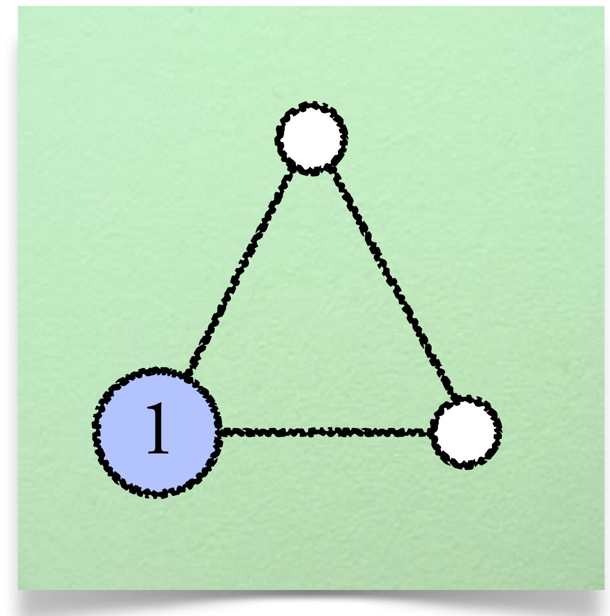
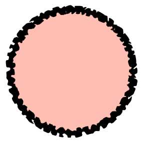
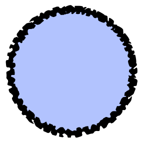


 = Spoiler

 = Duplicator

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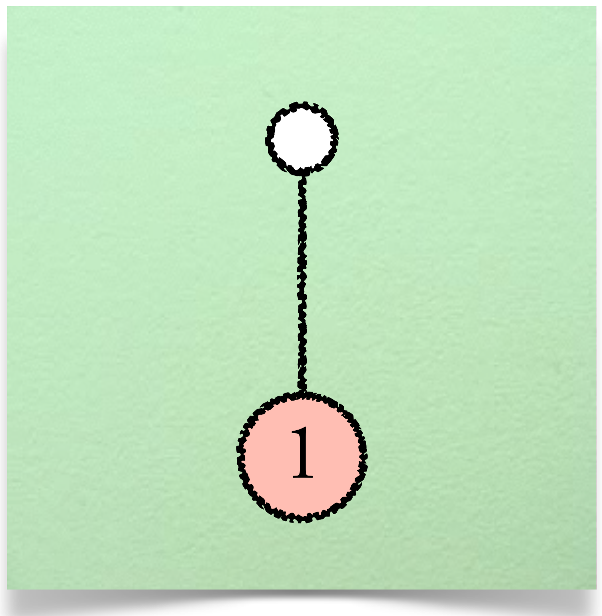

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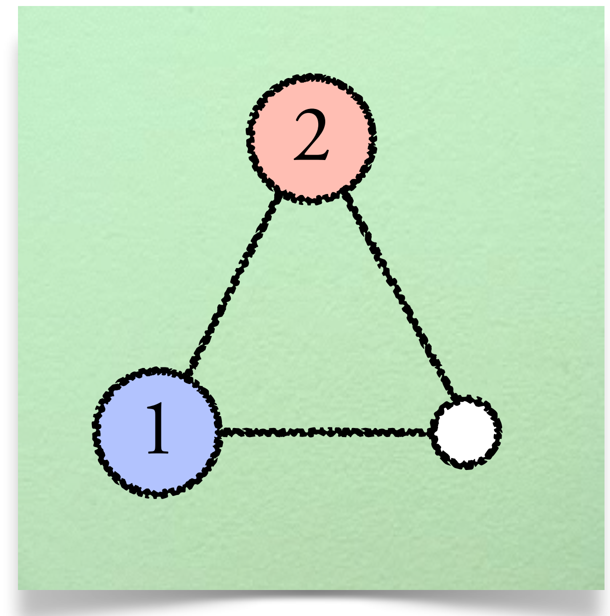
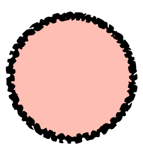
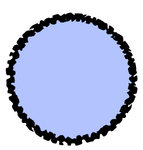


 = Spoiler

 = Duplicator

Ehrenfeucht-Fraïssé games

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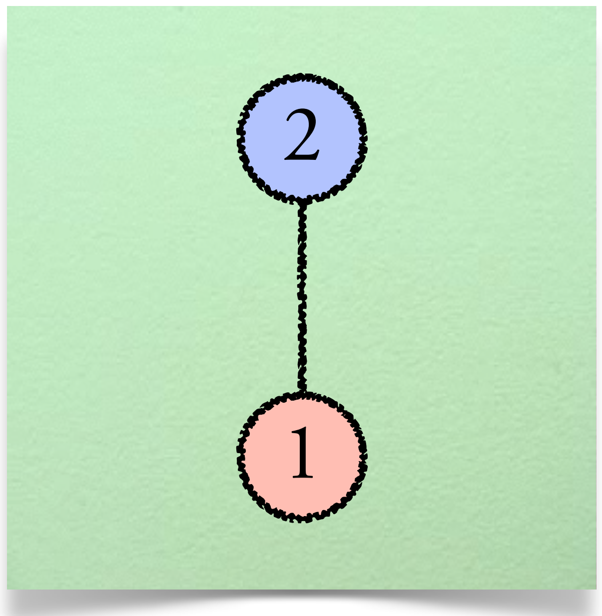

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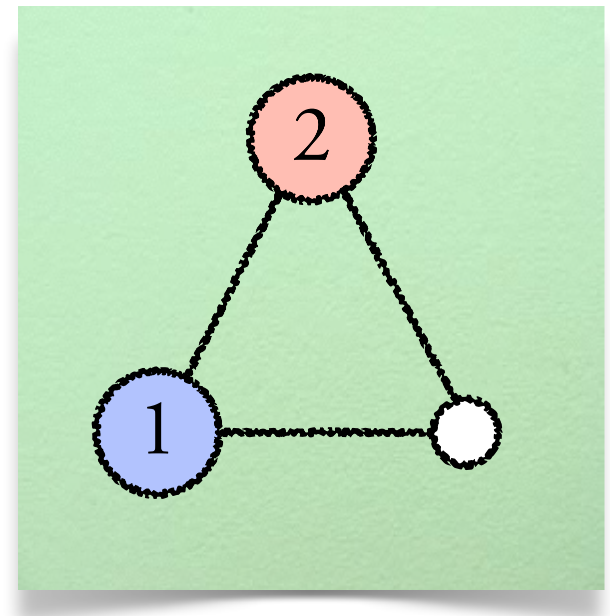
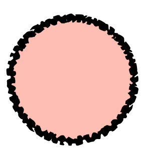
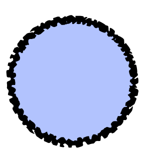


 = Spoiler

 = Duplicator

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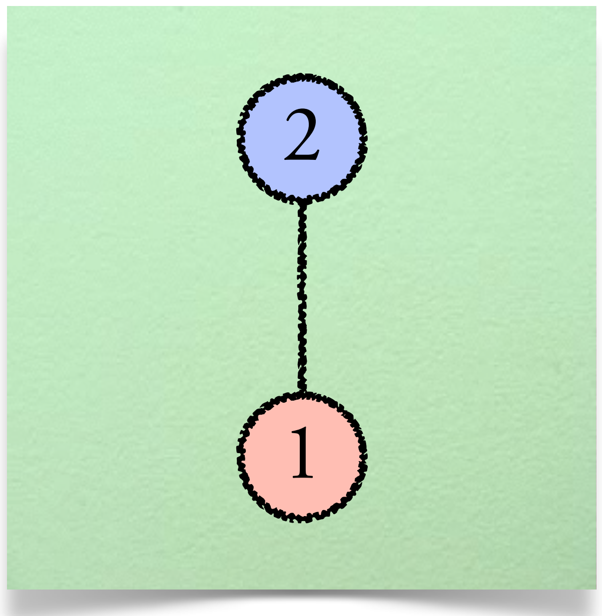

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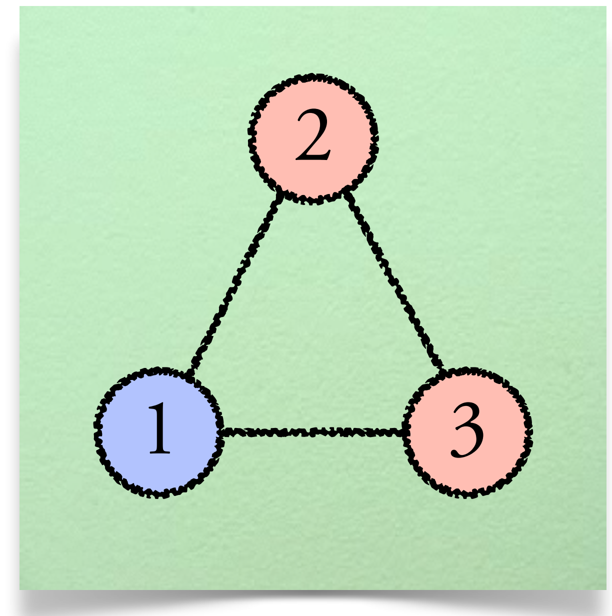
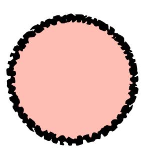
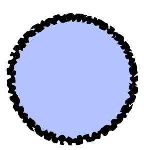


 = Spoiler

 = Duplicator

Ehrenfeucht-Fraïssé games

Question: Can Spoiler win in 3 rounds ?

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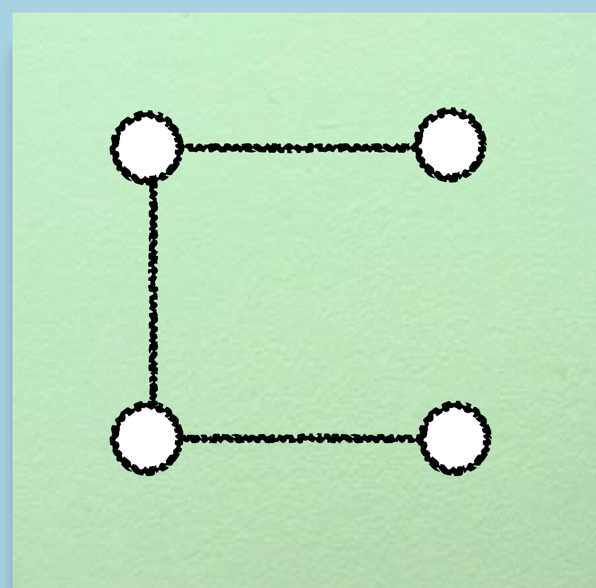

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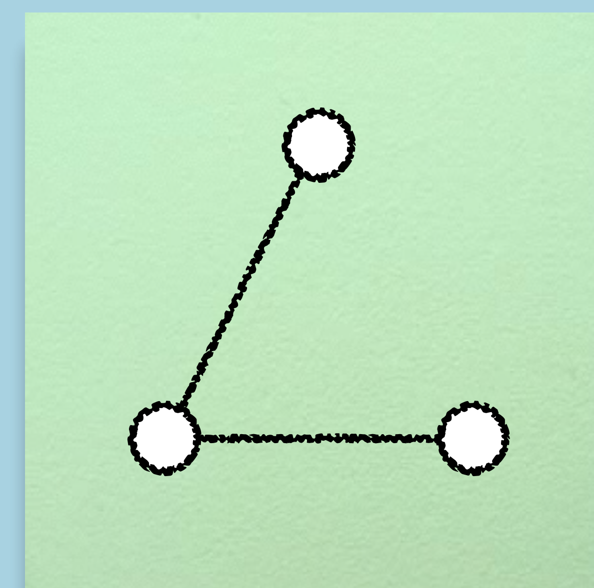

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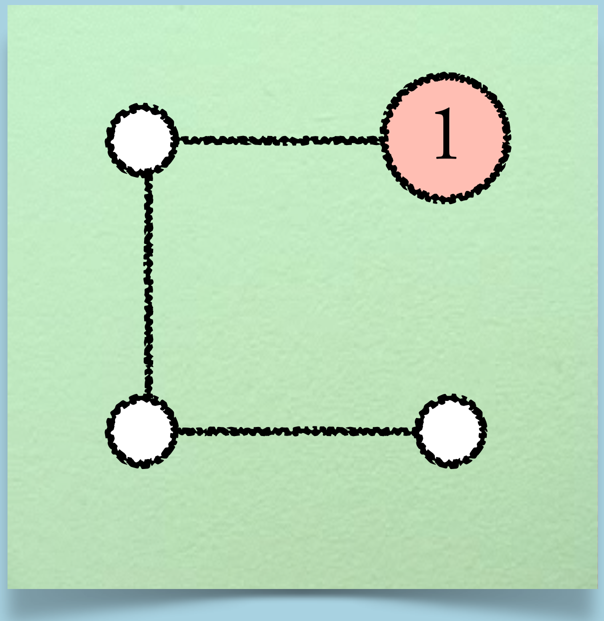

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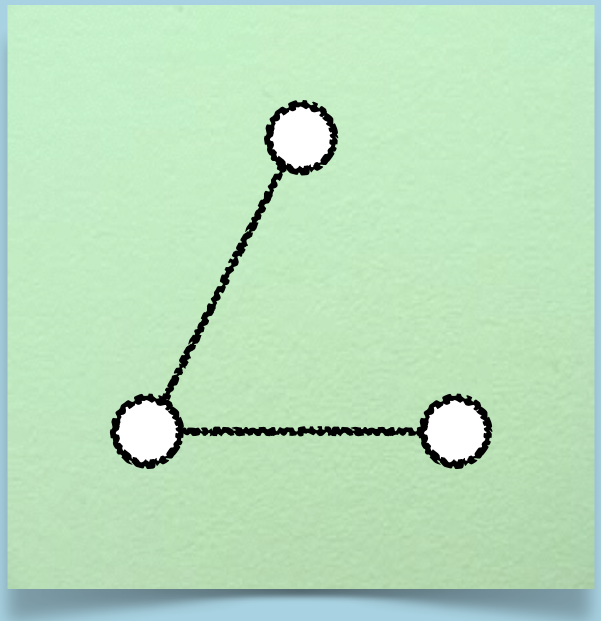

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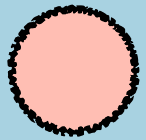

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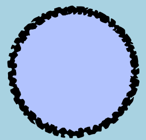

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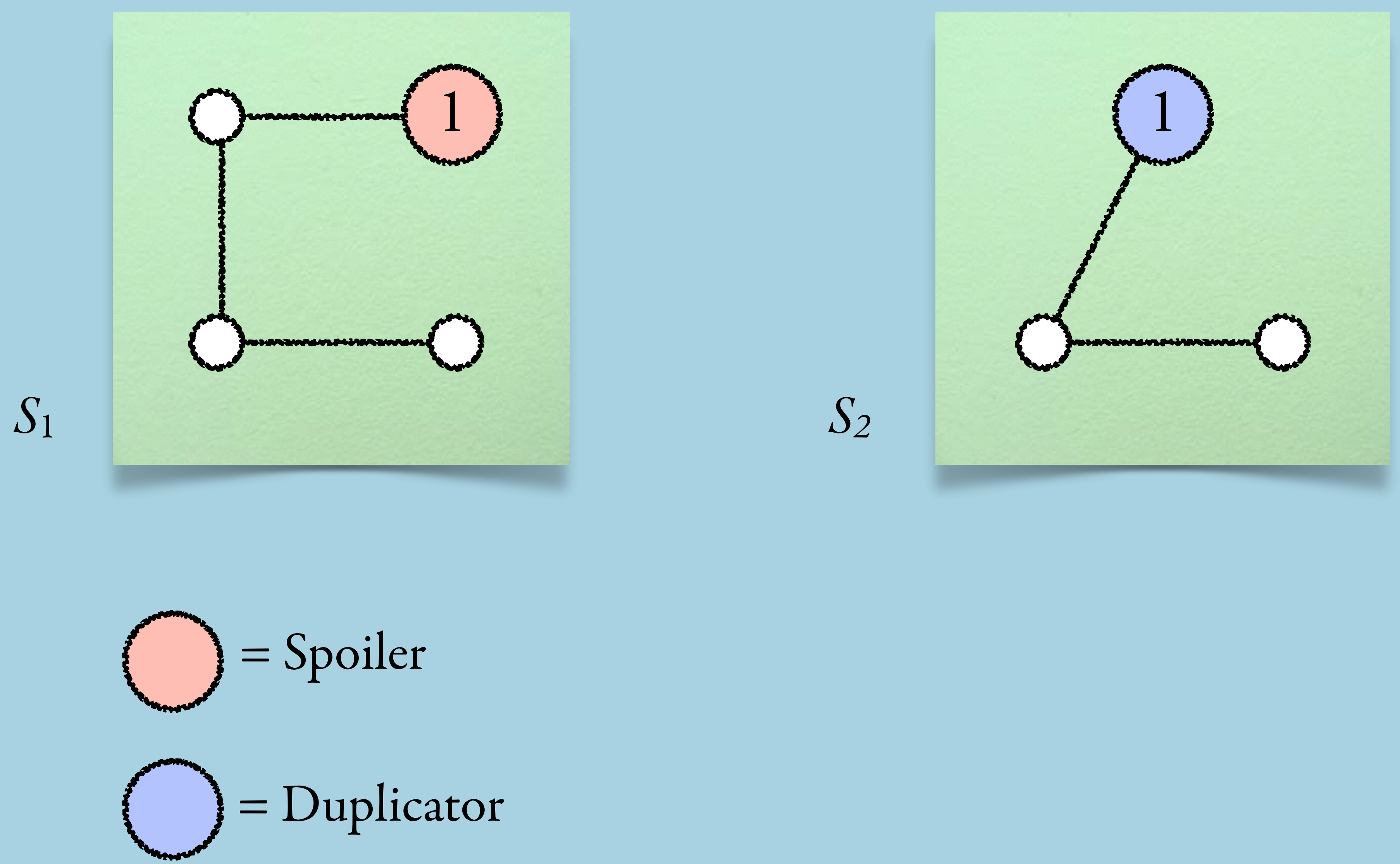

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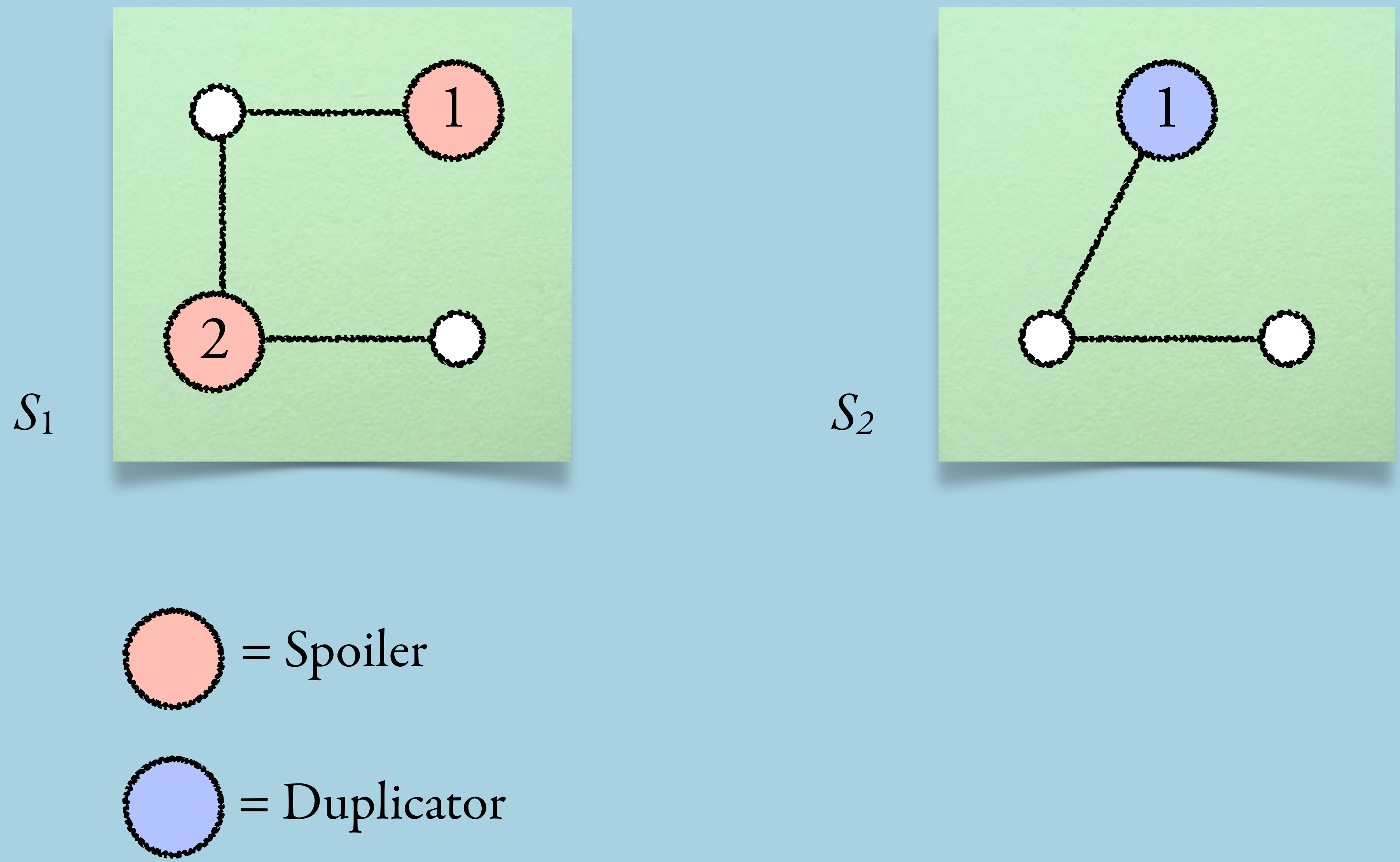

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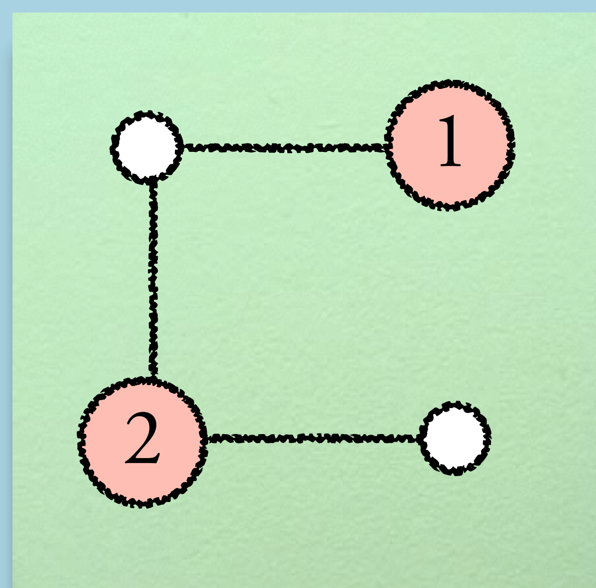

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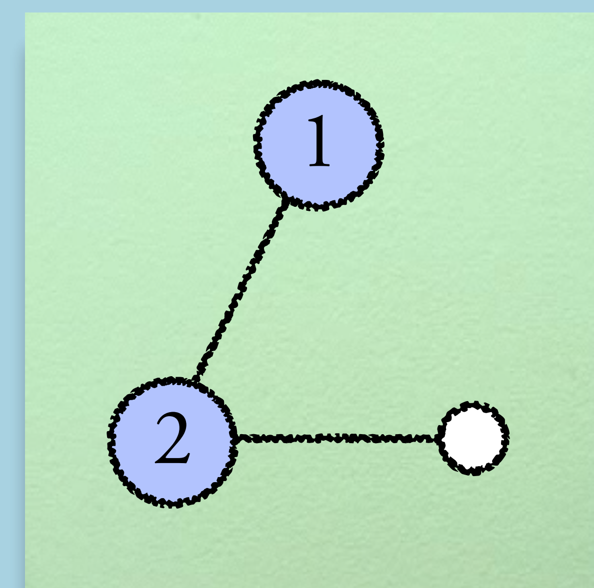

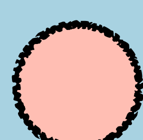

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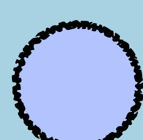

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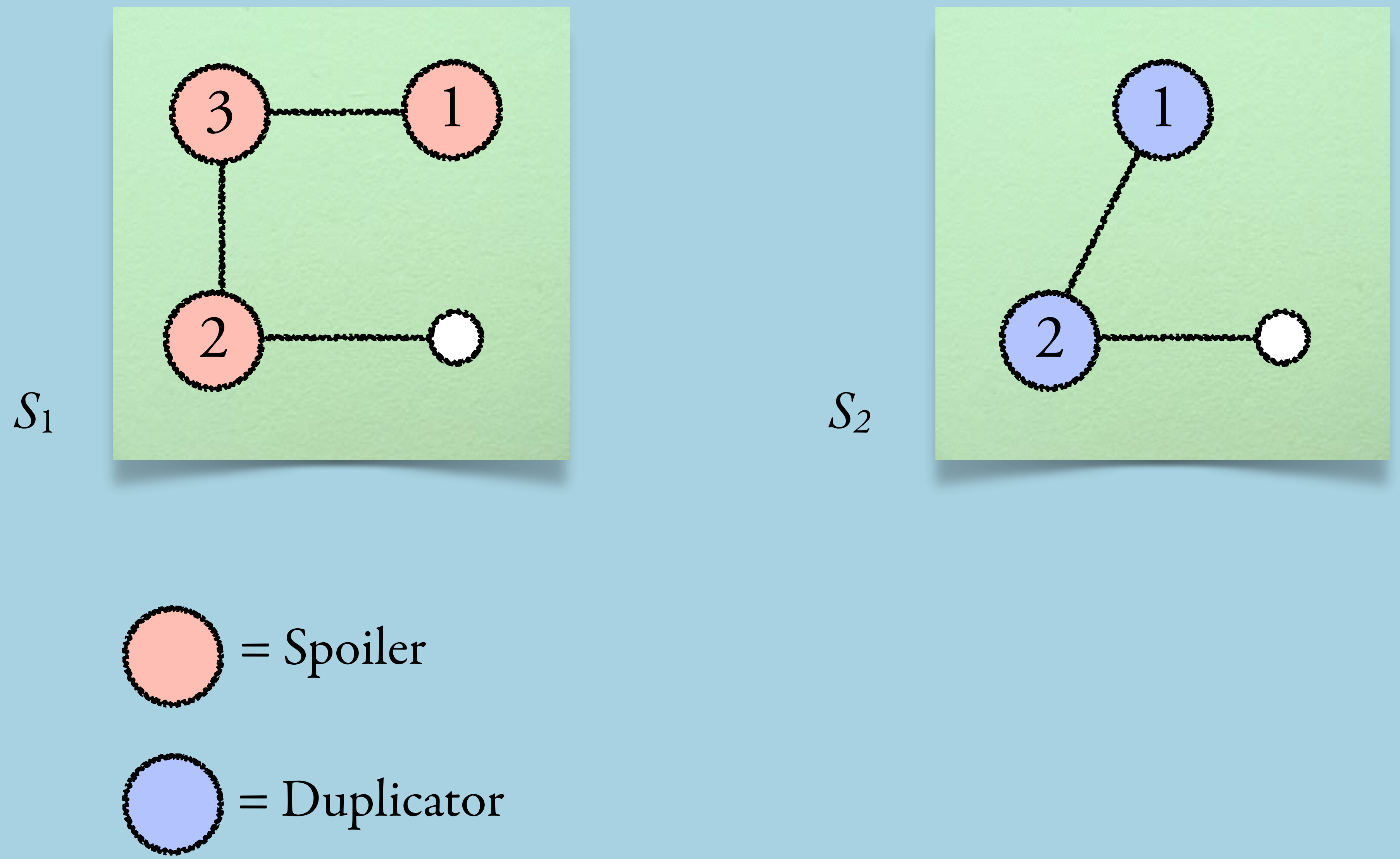

Ehrenfeucht-Fraïssé games

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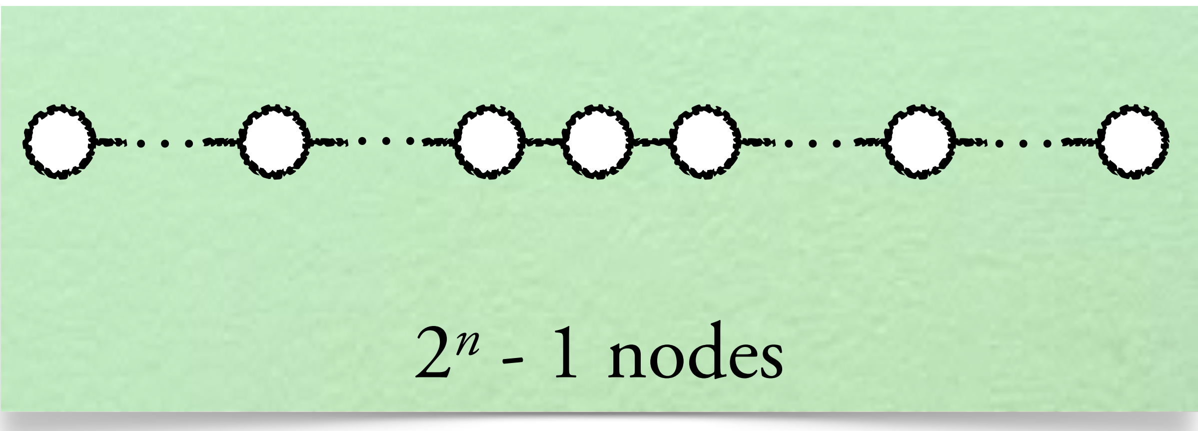

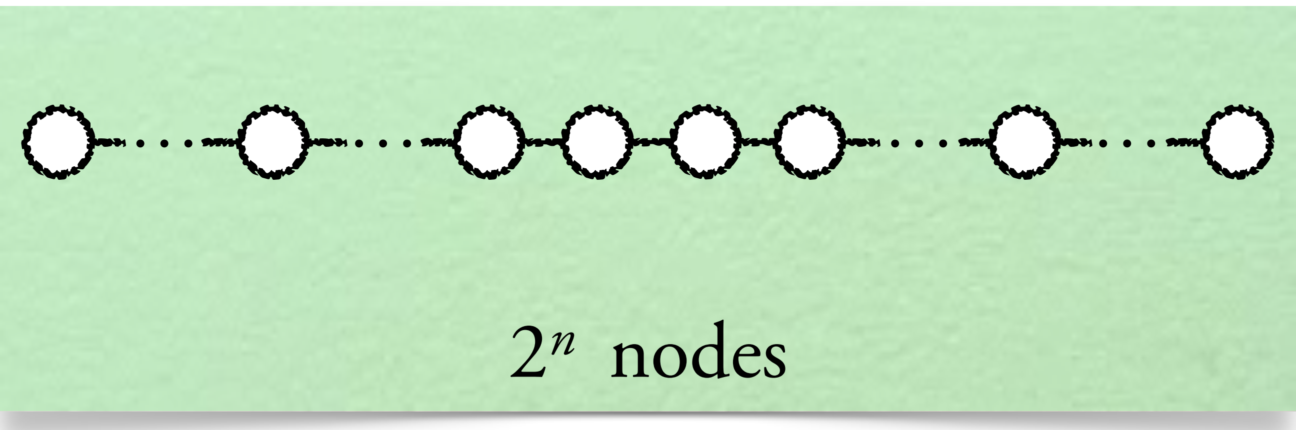

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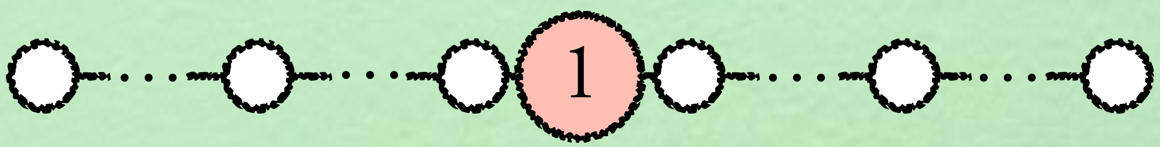
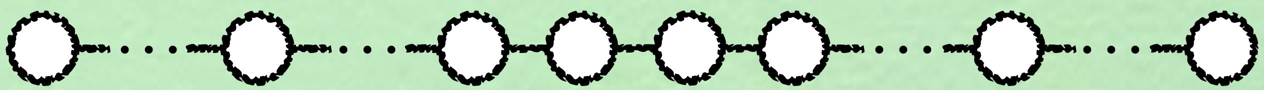


2^n - 1 nodes



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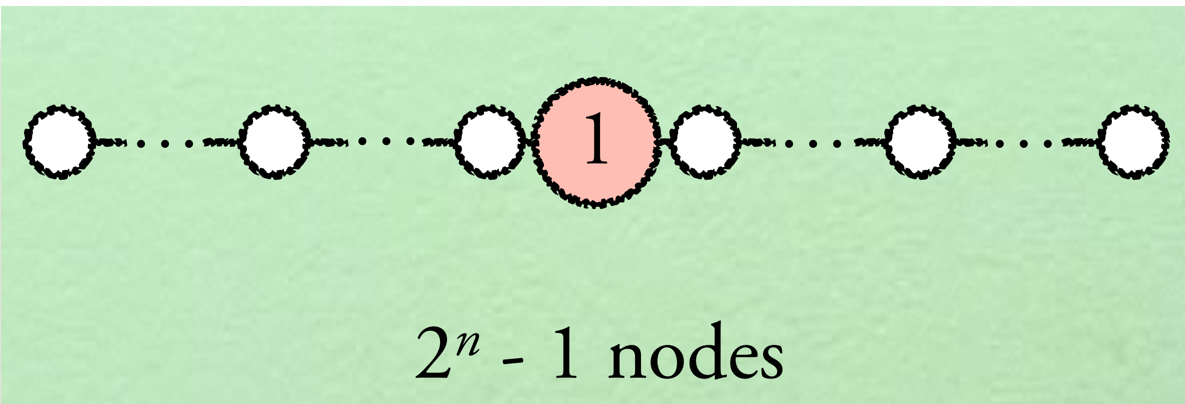

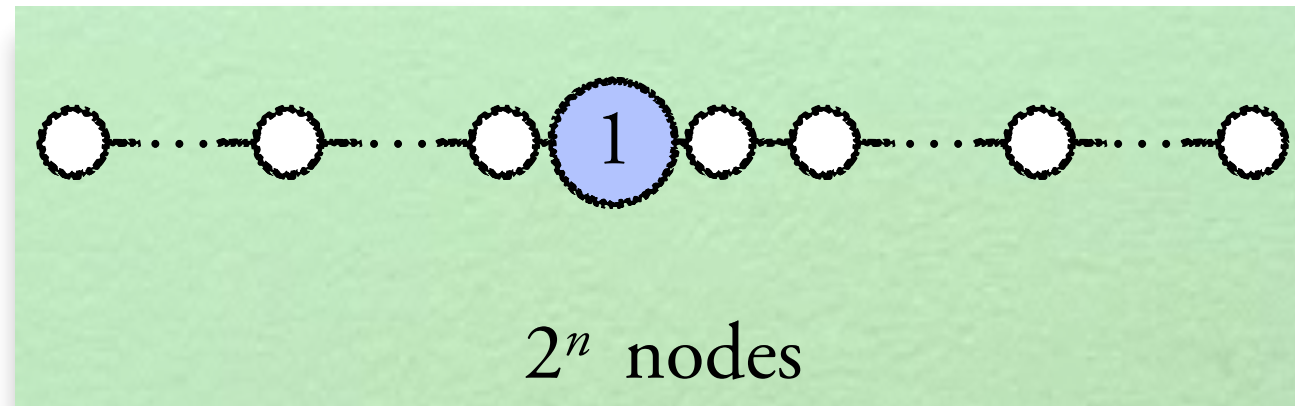

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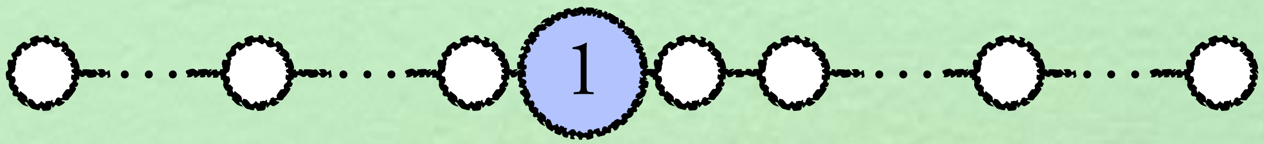

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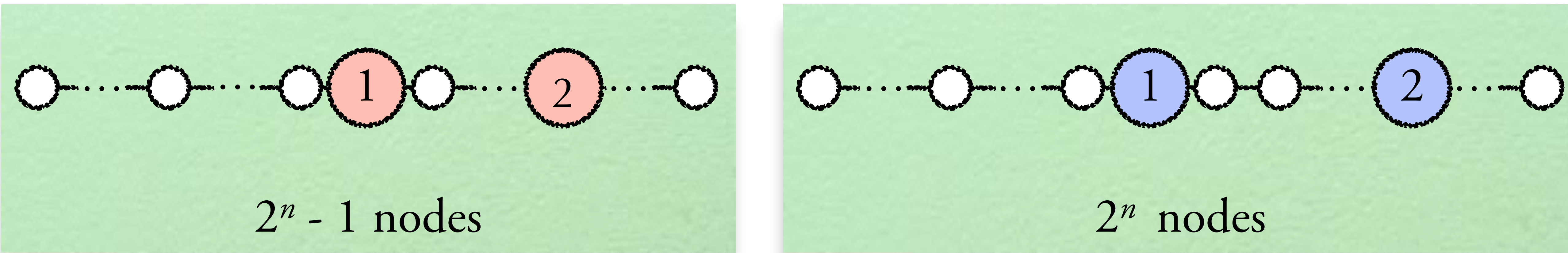

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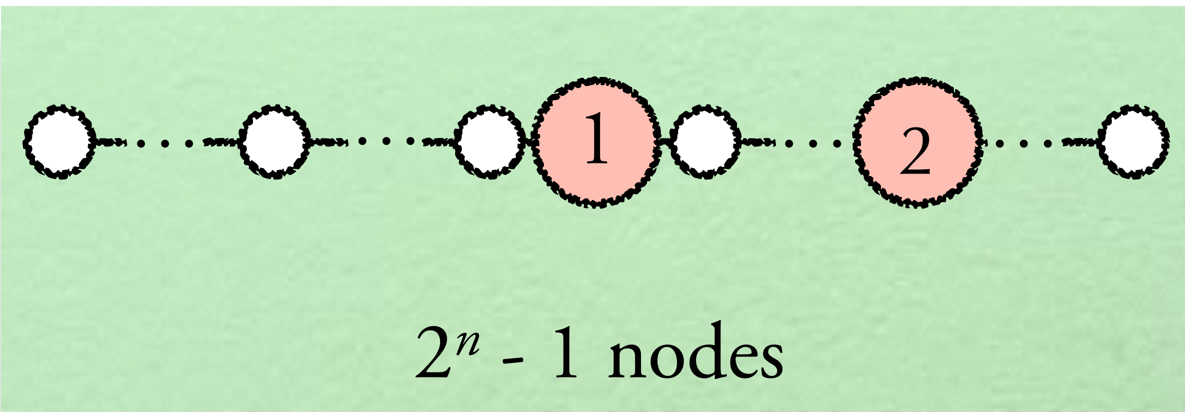

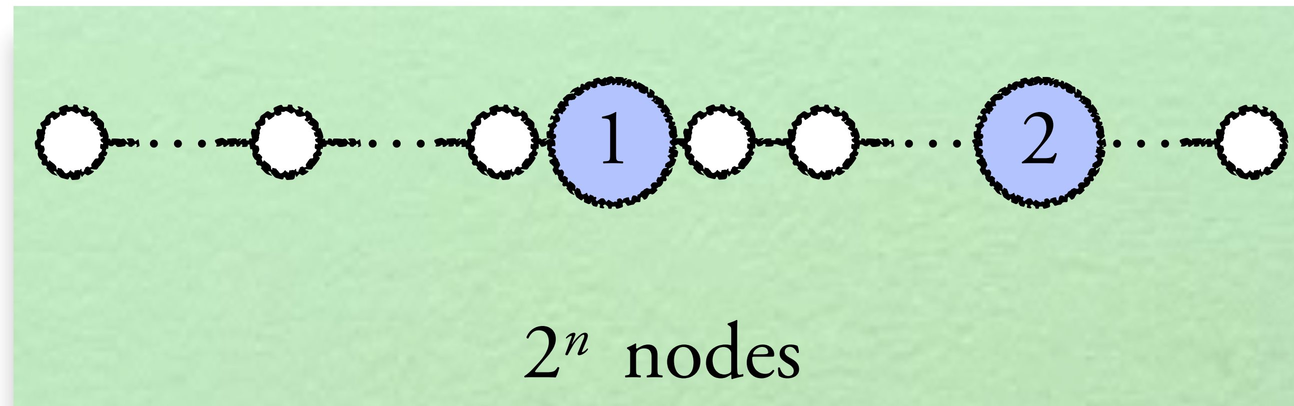

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Any idea?

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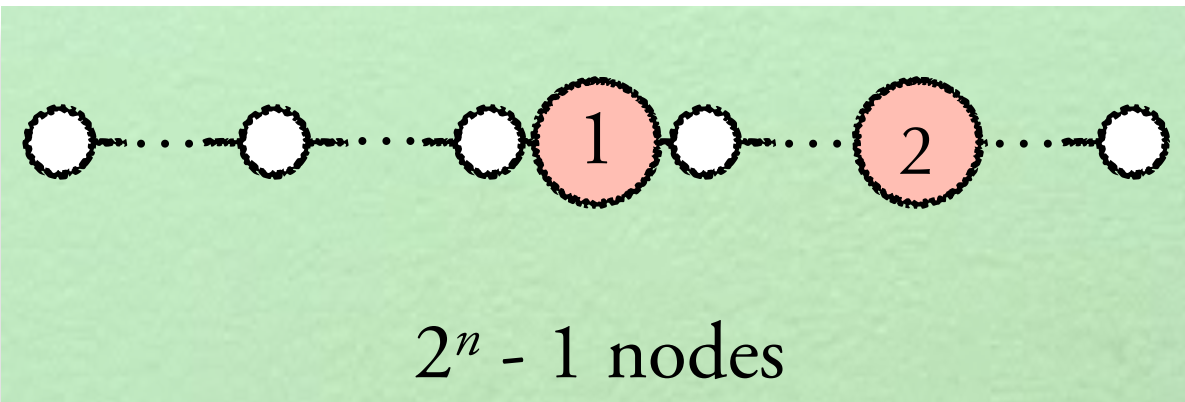

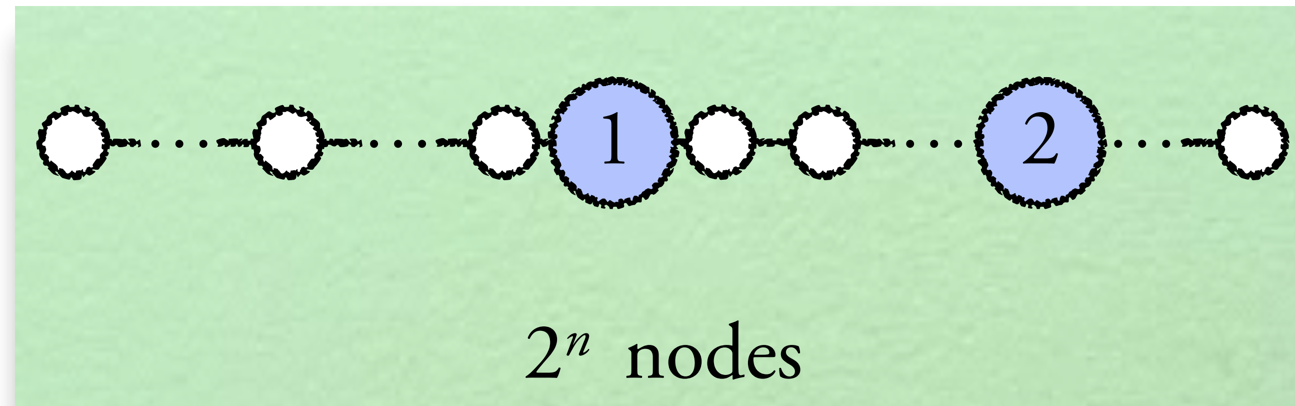

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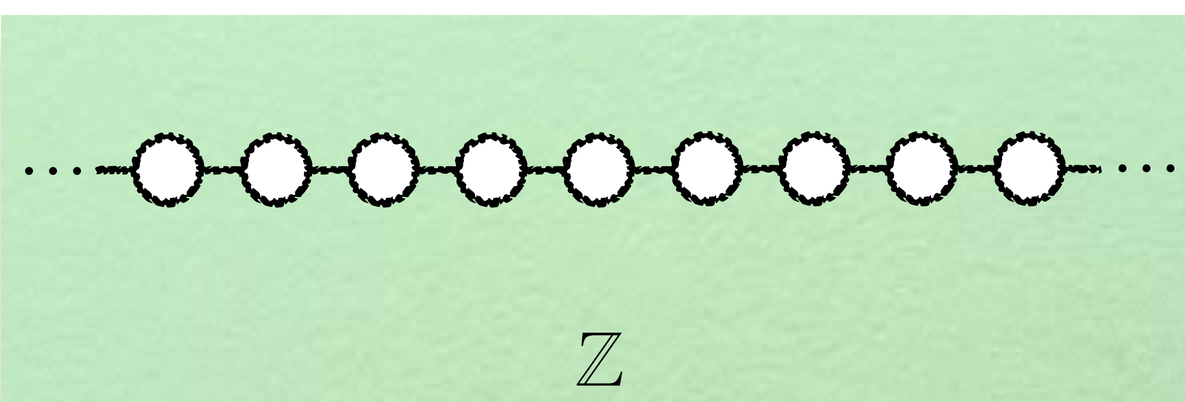

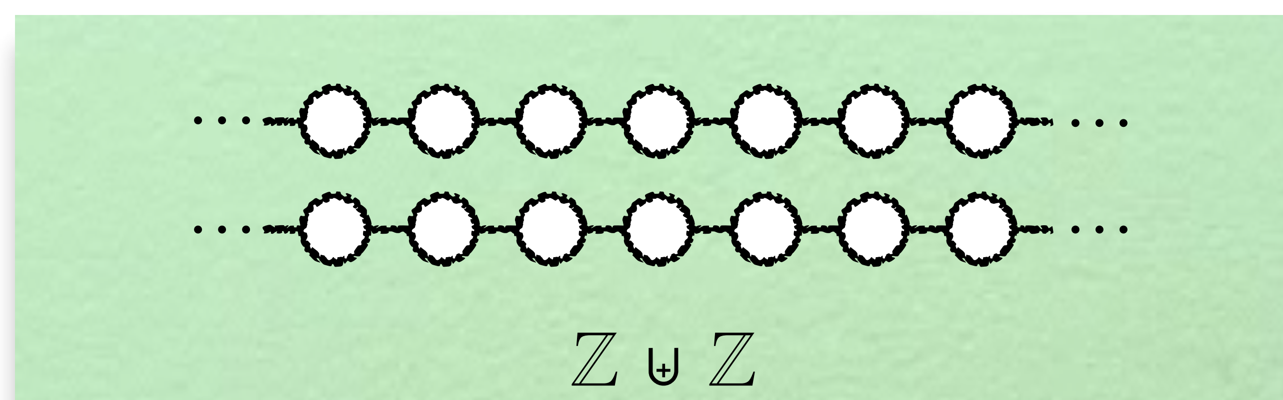

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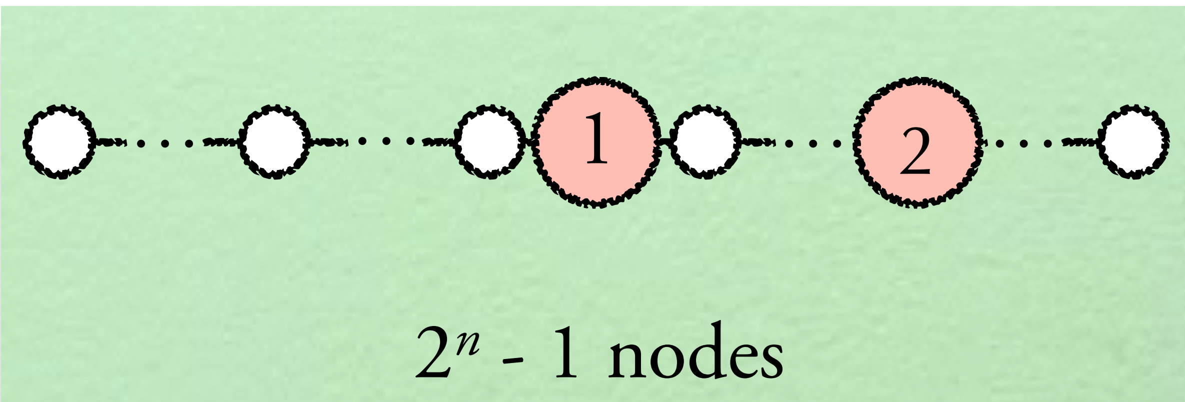

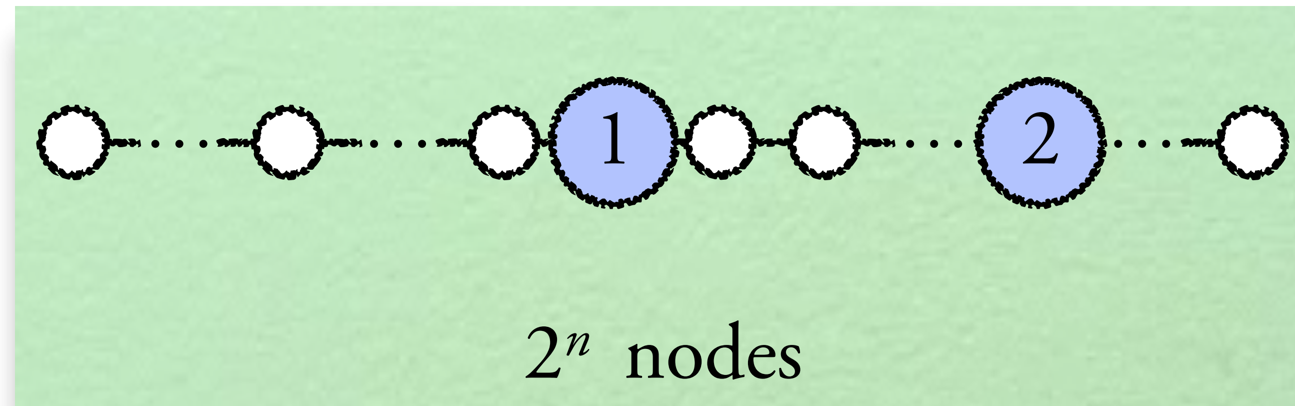

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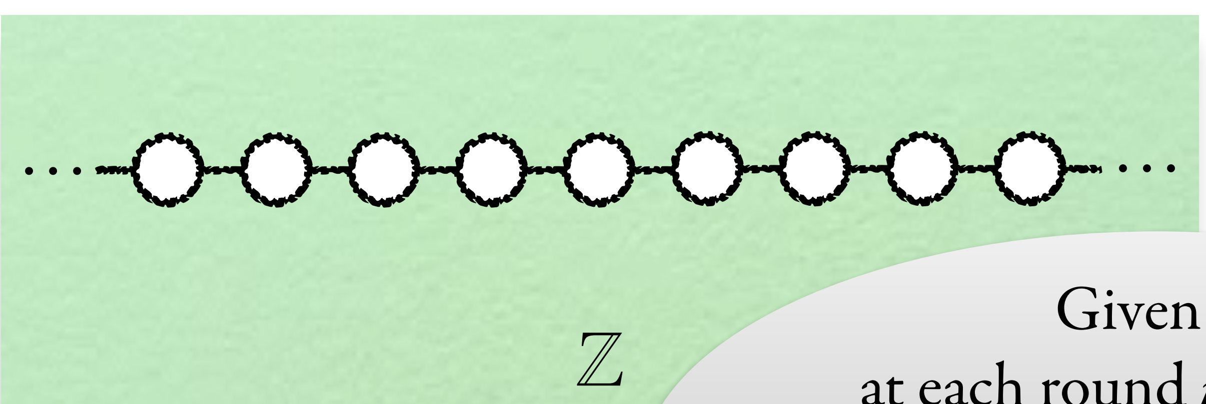

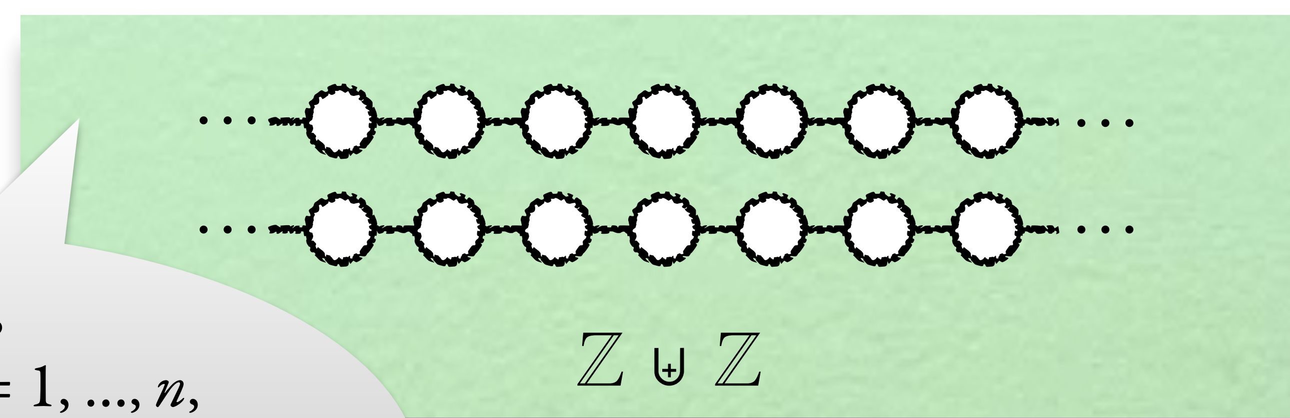

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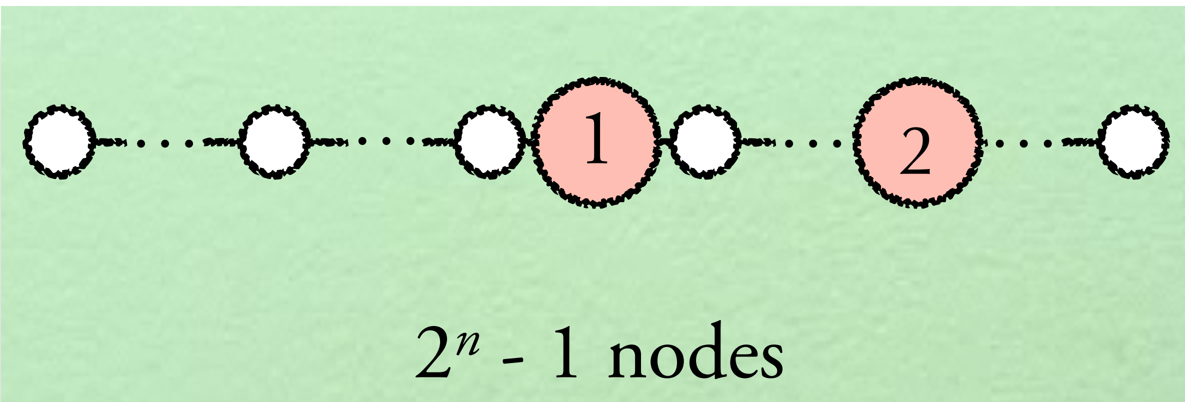

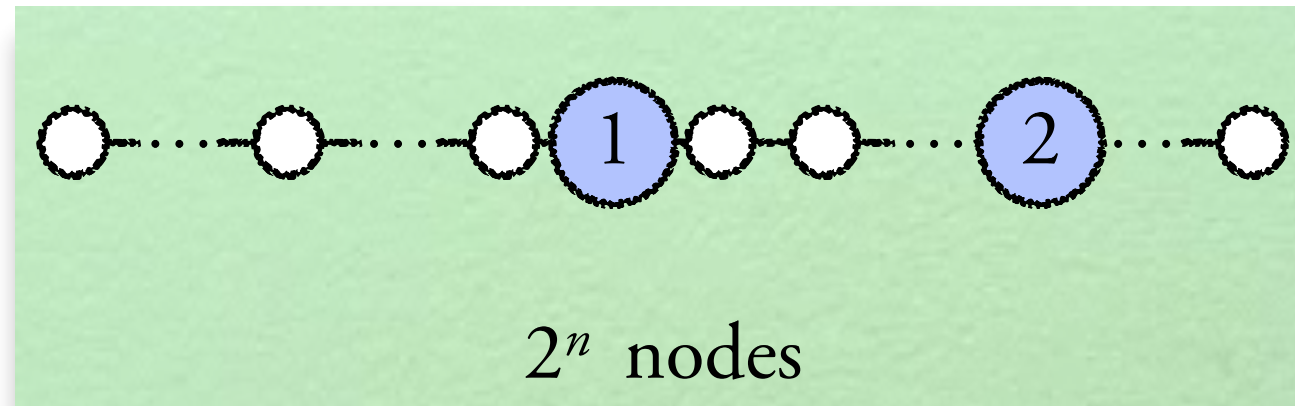

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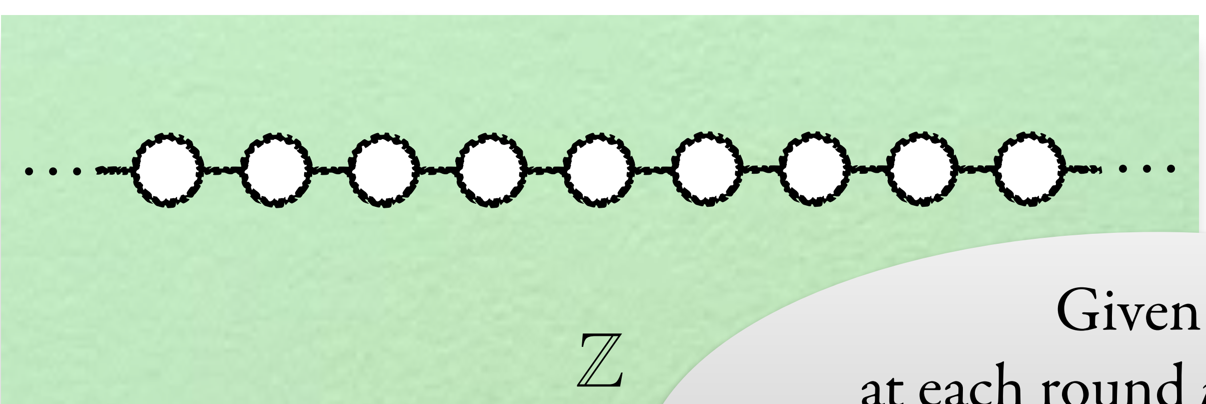

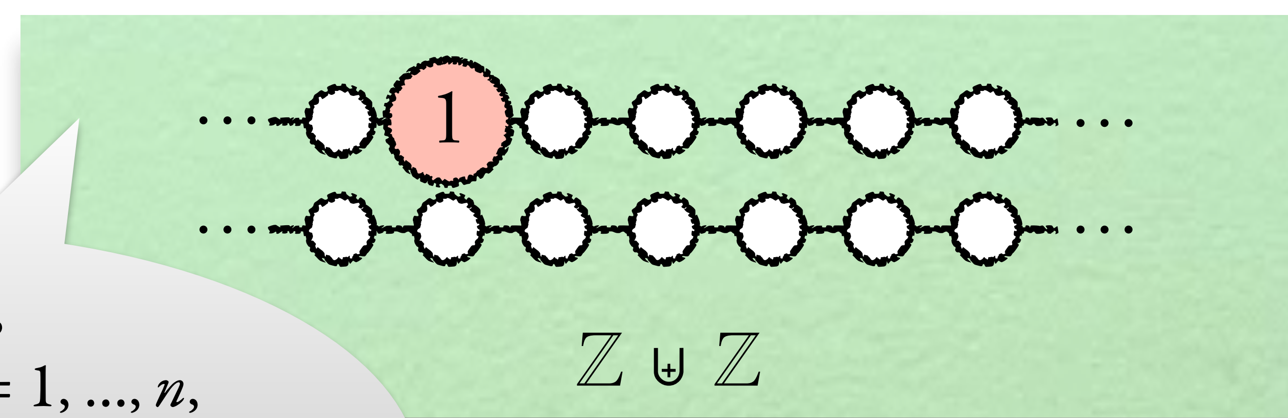

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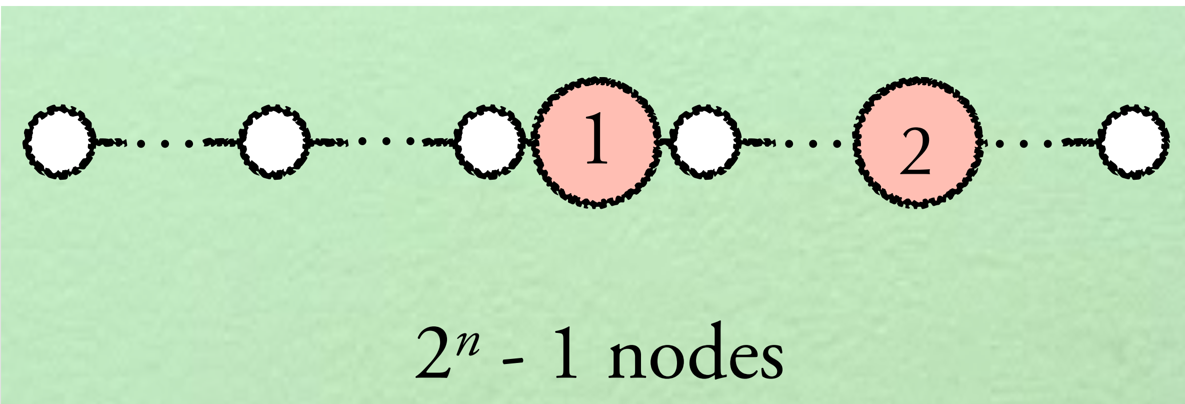

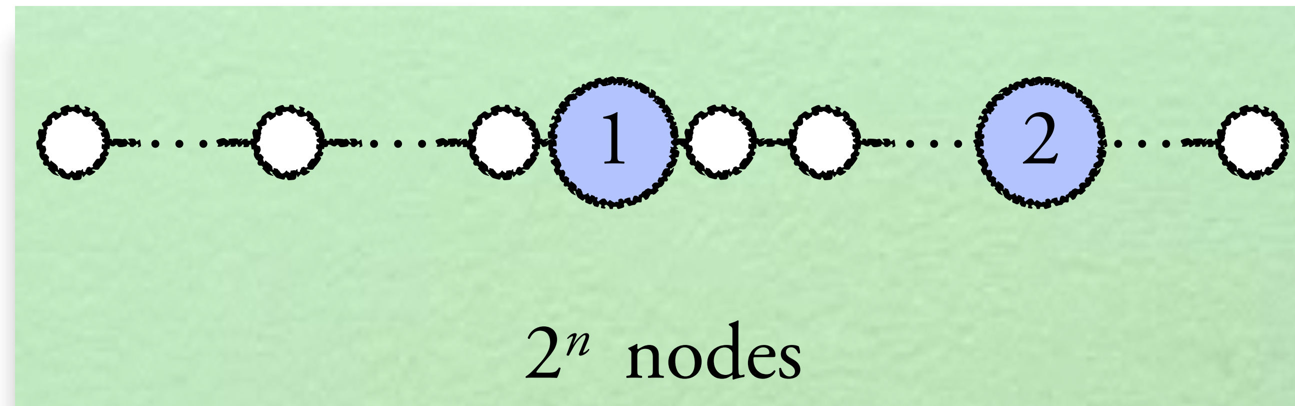

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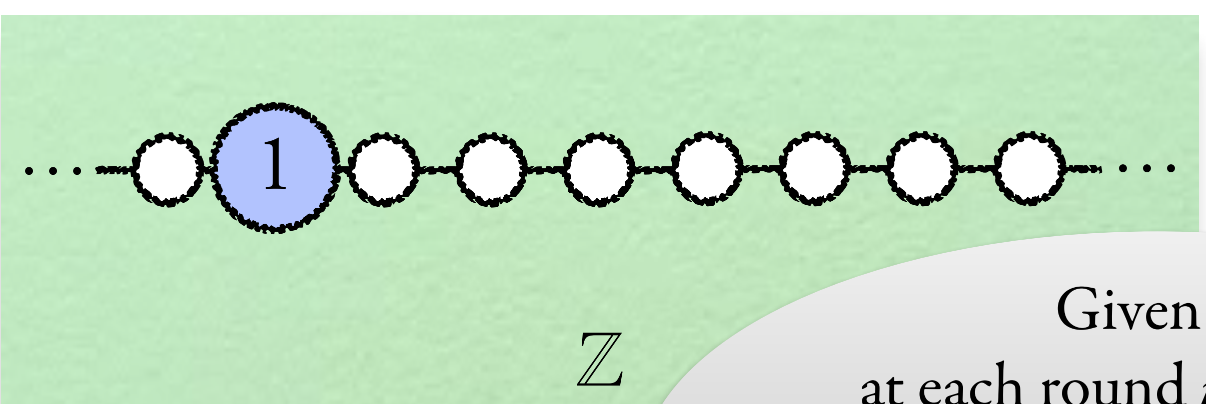

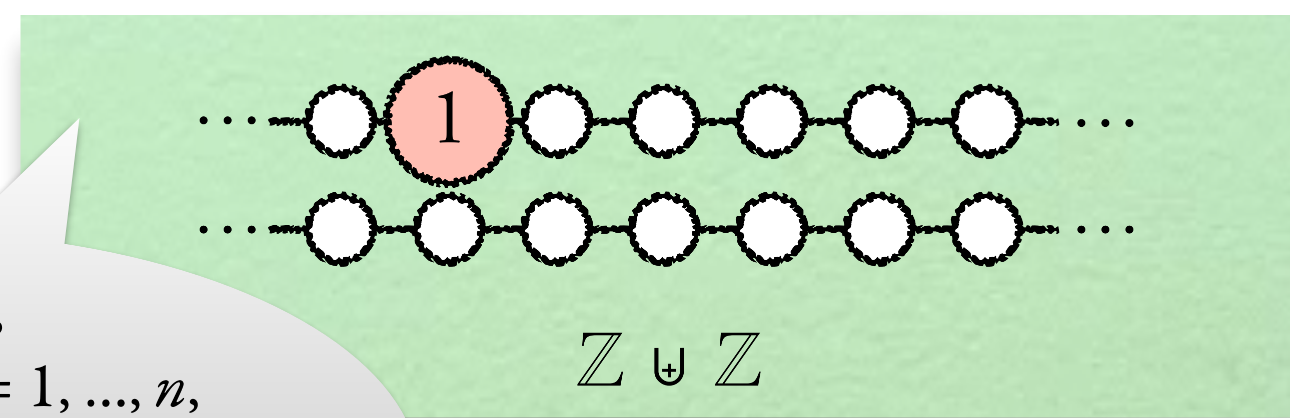

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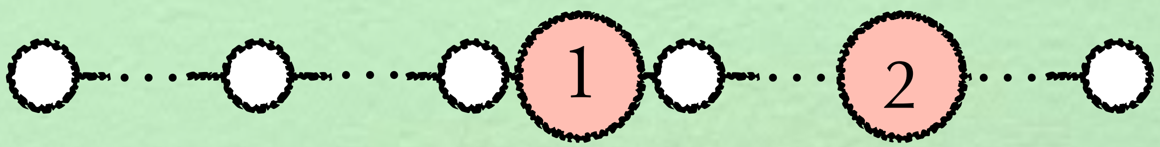

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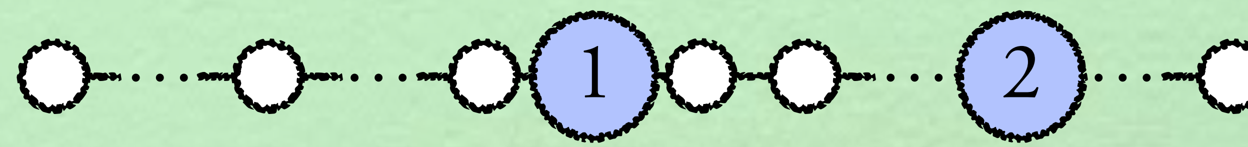

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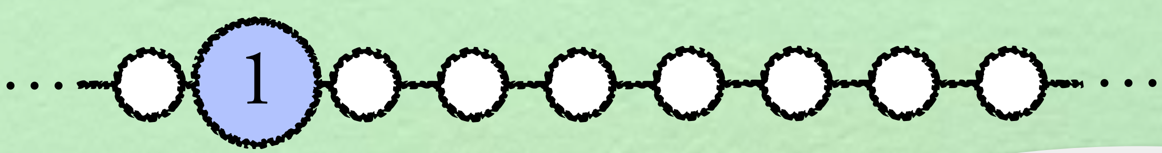
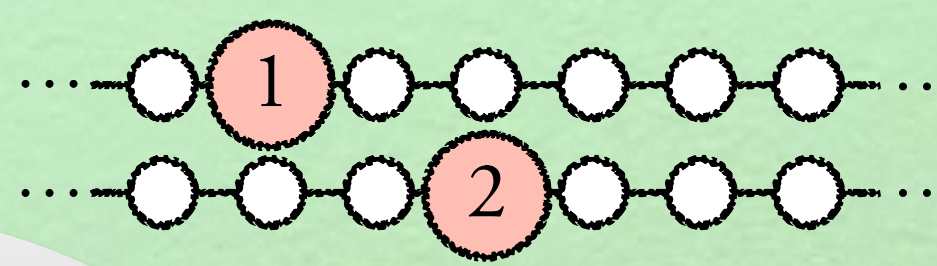


$\mathbb{Z}$



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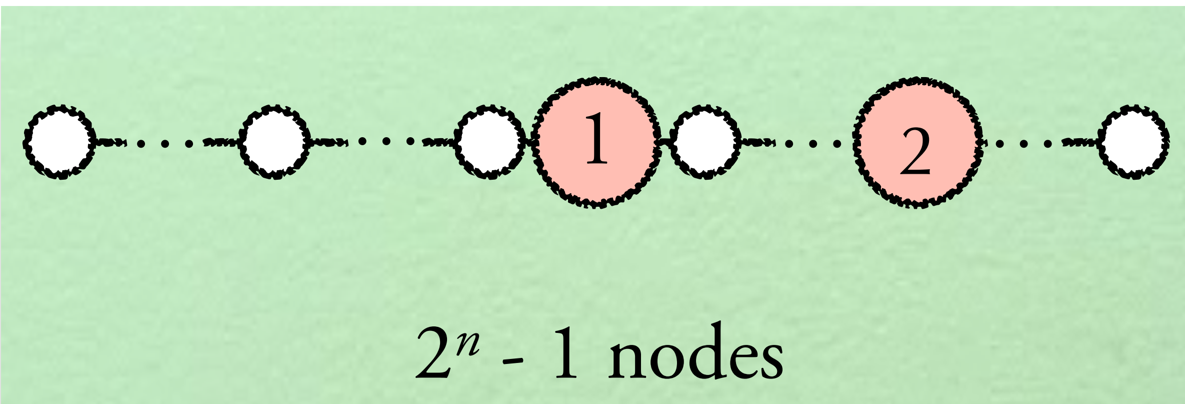

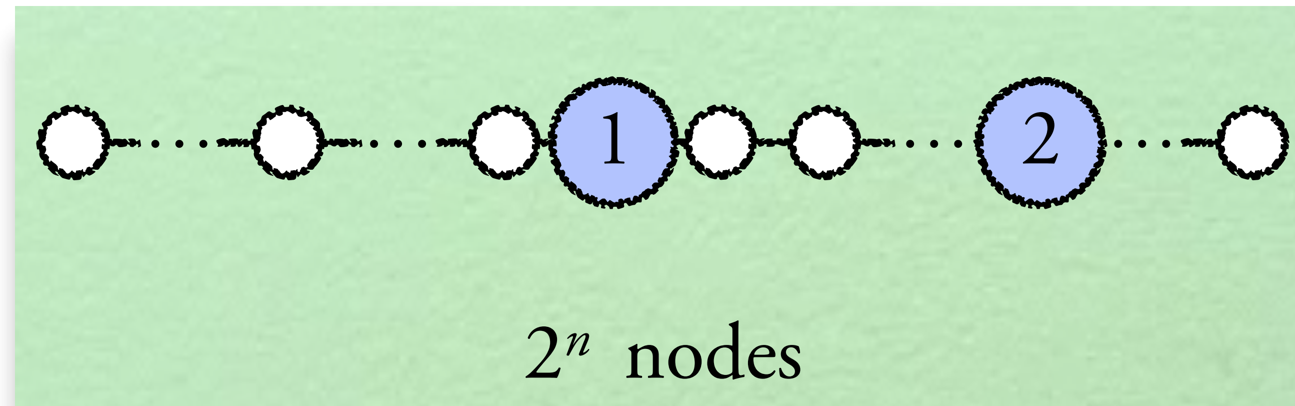

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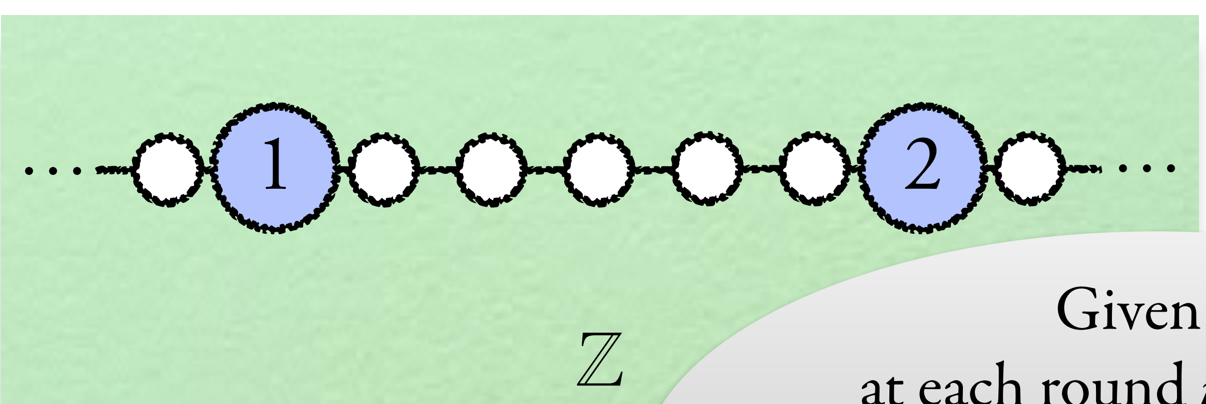

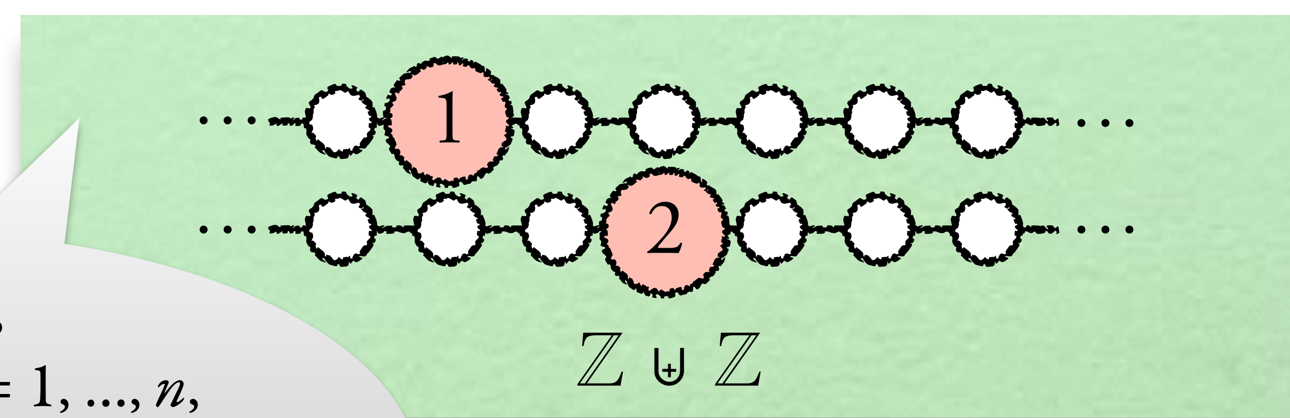

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Theorem. S_1 and S_2 are *n-equivalent* [Fraïssé '50, Ehrenfeucht '60]

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Duplicator has a strategy to survive n rounds in the EF game on S_1 and S_2.

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If **P** is a property and
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Then **P** is *not definable in FO*.

Proof: Suppose that **P** is defined by an FO sentence ϕ and ❄ holds.

From ❄ and the above theorem: $\forall n \ \exists S_1 \in \mathbf{P} \ \exists S_2 \notin \mathbf{P}$. S_1, S_2 are n-equivalent.

In particular, when n = *quantifier rank of* ϕ, $S_1 \models \phi$ iff $S_2 \models \phi$.

A contradiction!

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A new game to evaluate formulas....

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push negations inside:

$$\neg\forall\phi \rightsquigarrow \exists\neg\phi$$

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Whether $S \vDash \phi$ can be decided by a ***new game*** between two players, **True** and **False**:

- $\phi = E(x,y)$ → **True** wins if nodes marked x and y are connected by an edge, otherwise he loses
- $\phi = \exists x\ \phi'(x)$ → **True** moves by marking a node x in S, the game continues with ϕ'
- $\phi = \forall y\ \phi'(y)$ → **False** moves by marking a node y in S, the game continues with ϕ'
- $\phi = \phi_1 \vee \phi_2$ → **True** moves by choosing ϕ_1 or ϕ_2, the game continues with what he chose
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Whether $S \models \phi$ can be decided by a ***new game*** between two players, True and False:

- $\phi = E(x,y)$ → True wins if nodes marked x and y are connected by an edge, otherwise he loses
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Lemma. $S \models \phi$ iff True wins the semantics game.

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Turn winning strategy for True in S_1 into winning strategy for True in S_2 ....

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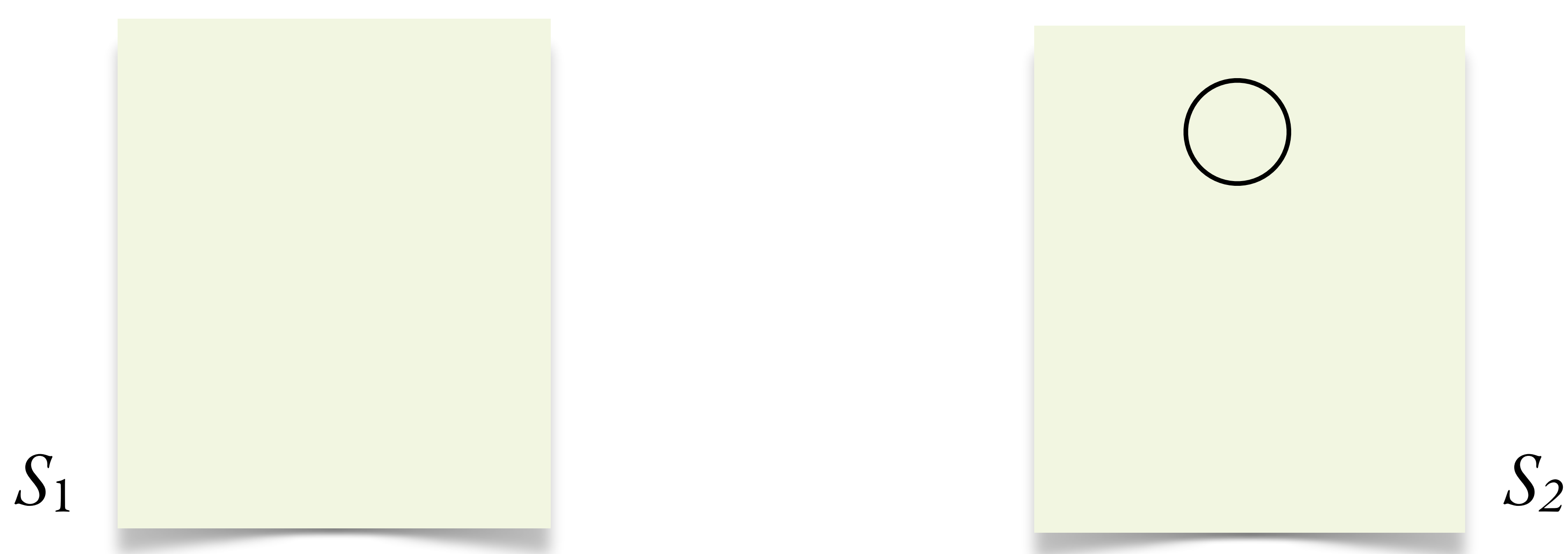

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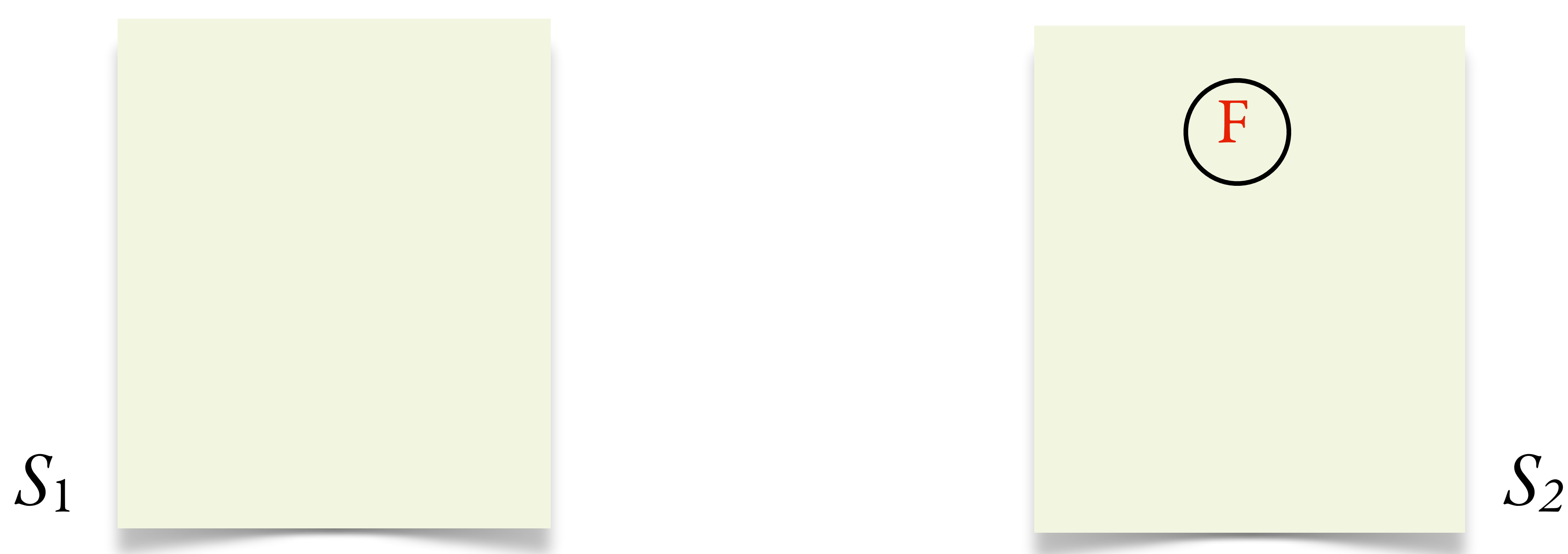

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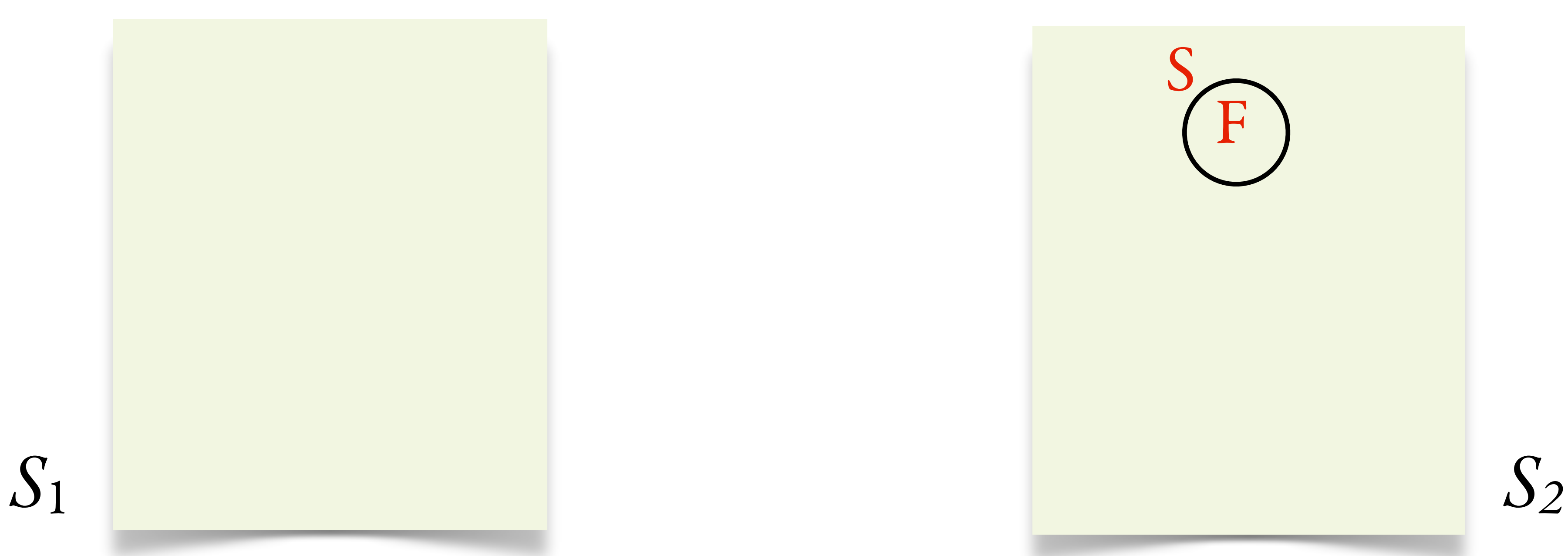

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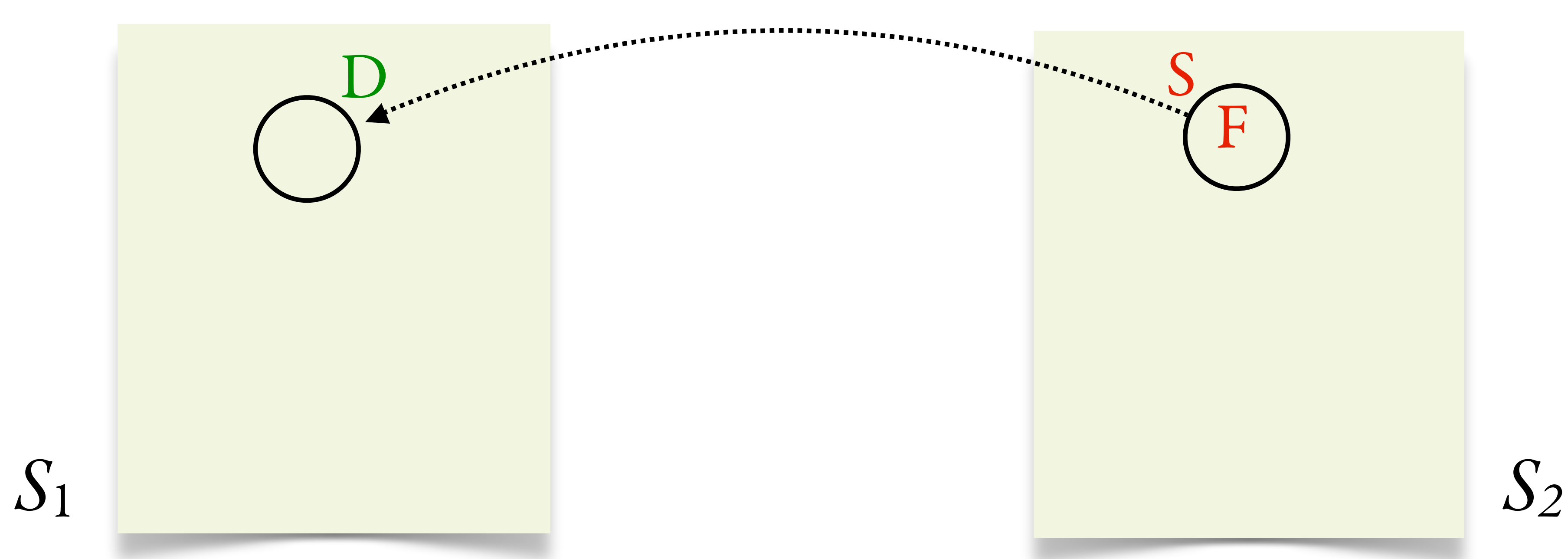

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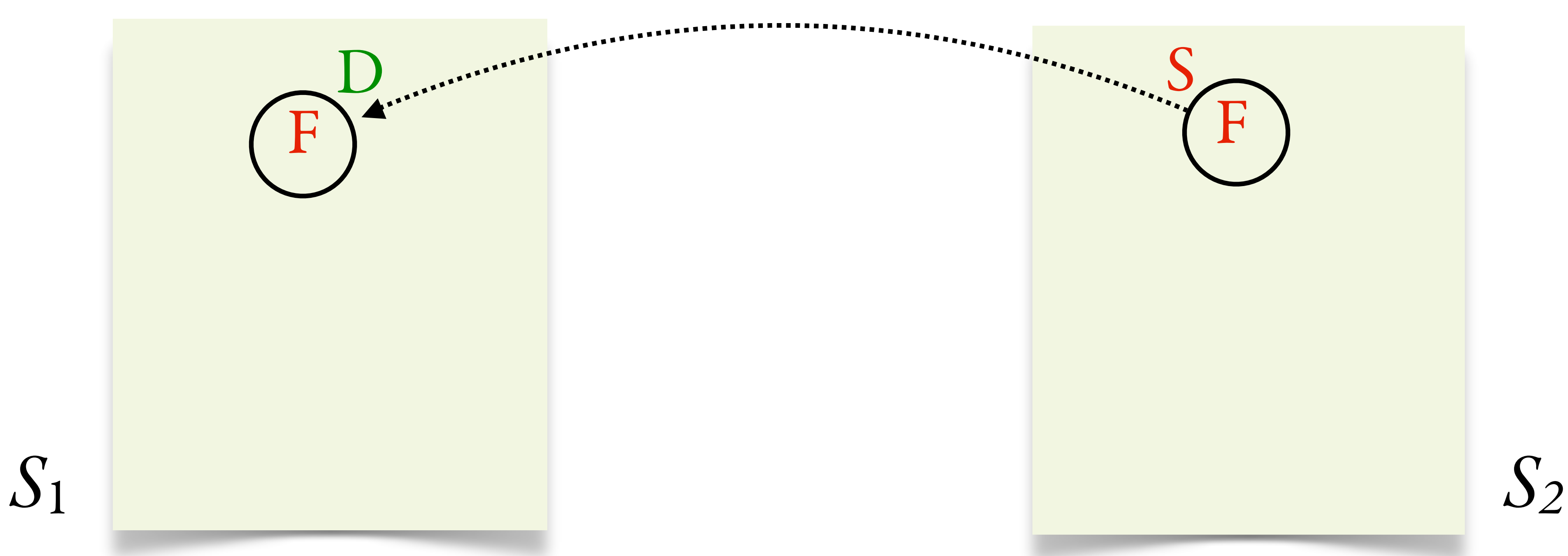

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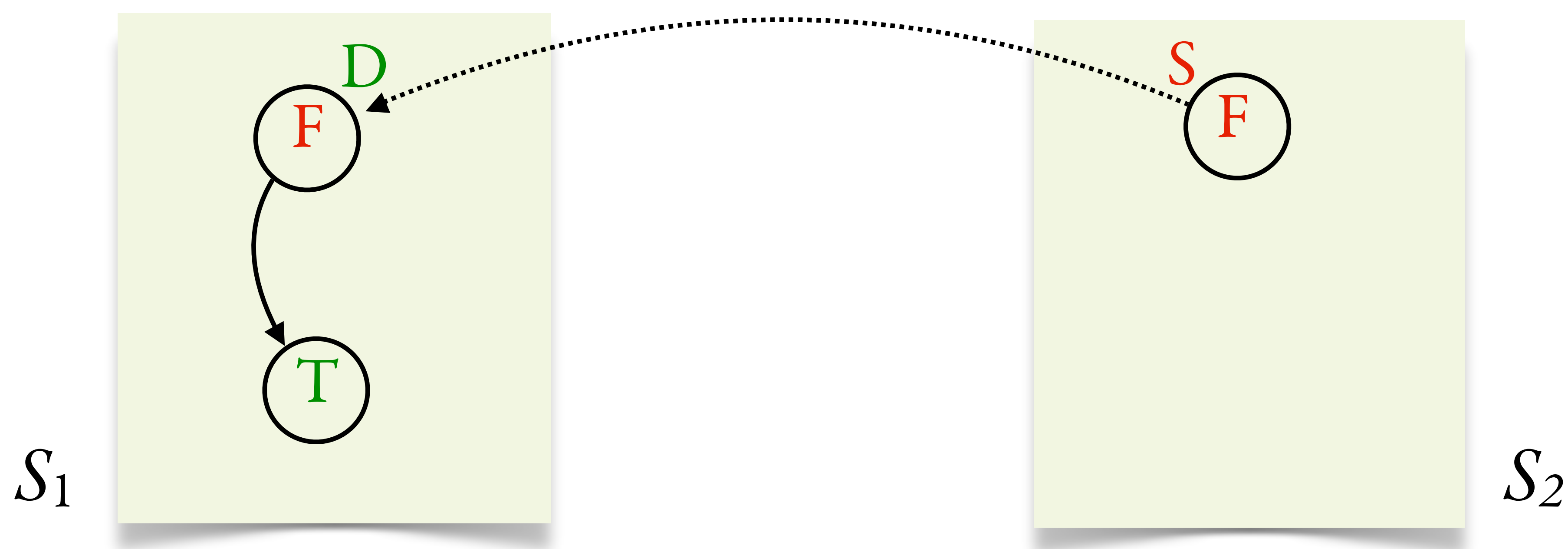

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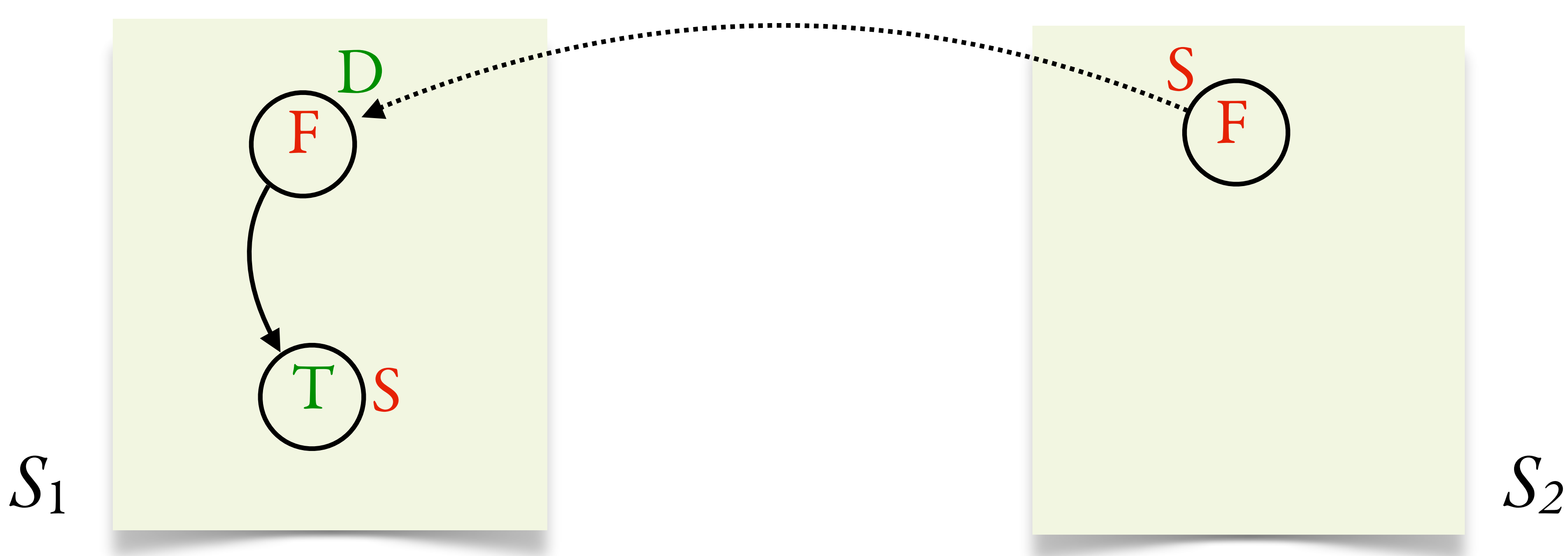

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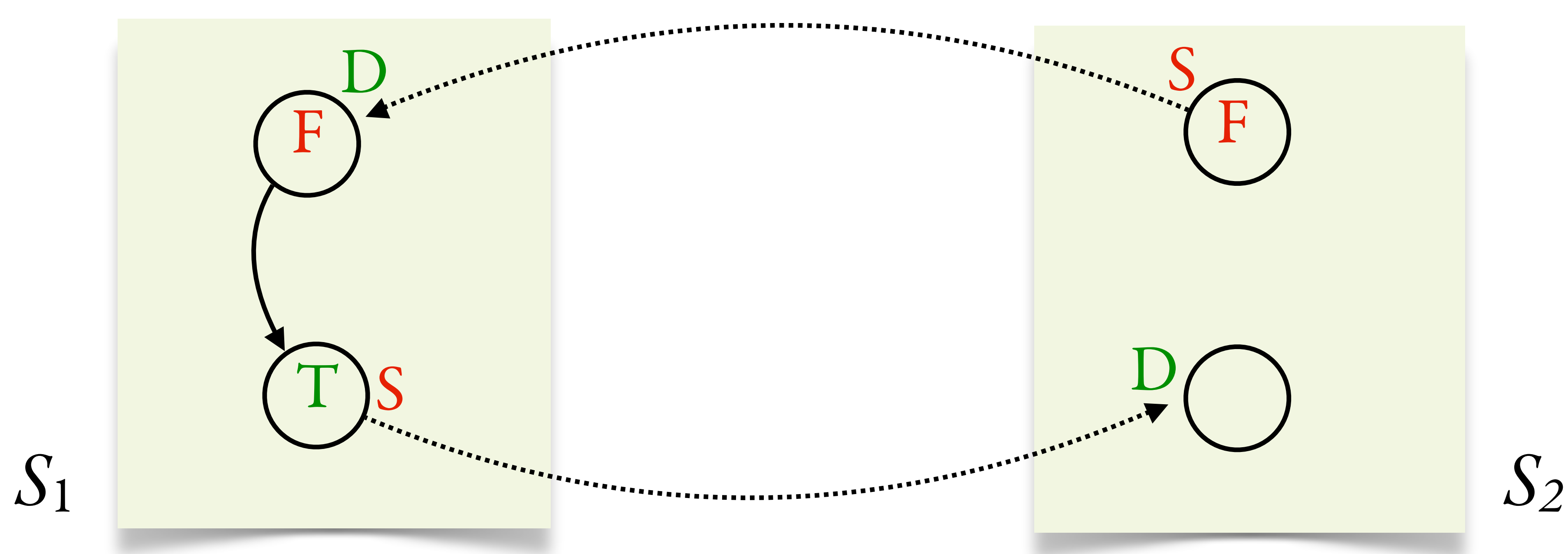
Proof ideas for the if-direction (from Duplicator's winning strategy to n-equivalence)

Consider ϕ with quantifier rank n.

Suppose $S_1 \models \phi$ and Duplicator survives n rounds on S_1, S_2.

We need to prove that $S_2 \models \phi$.



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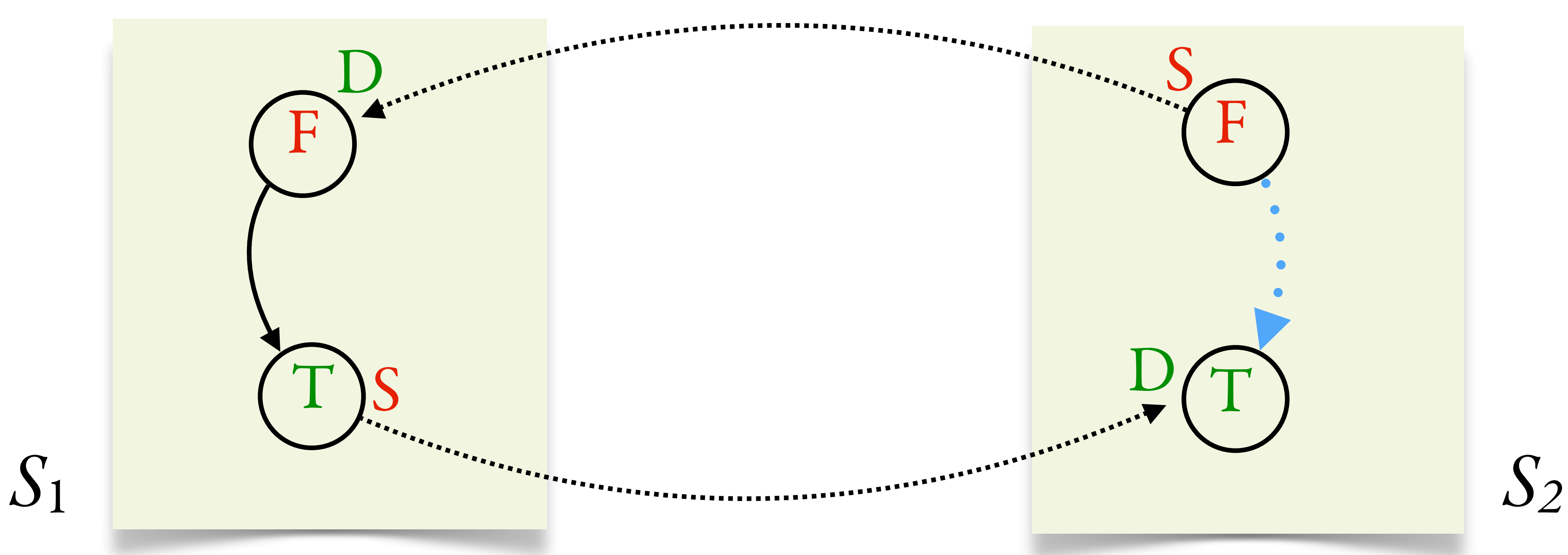

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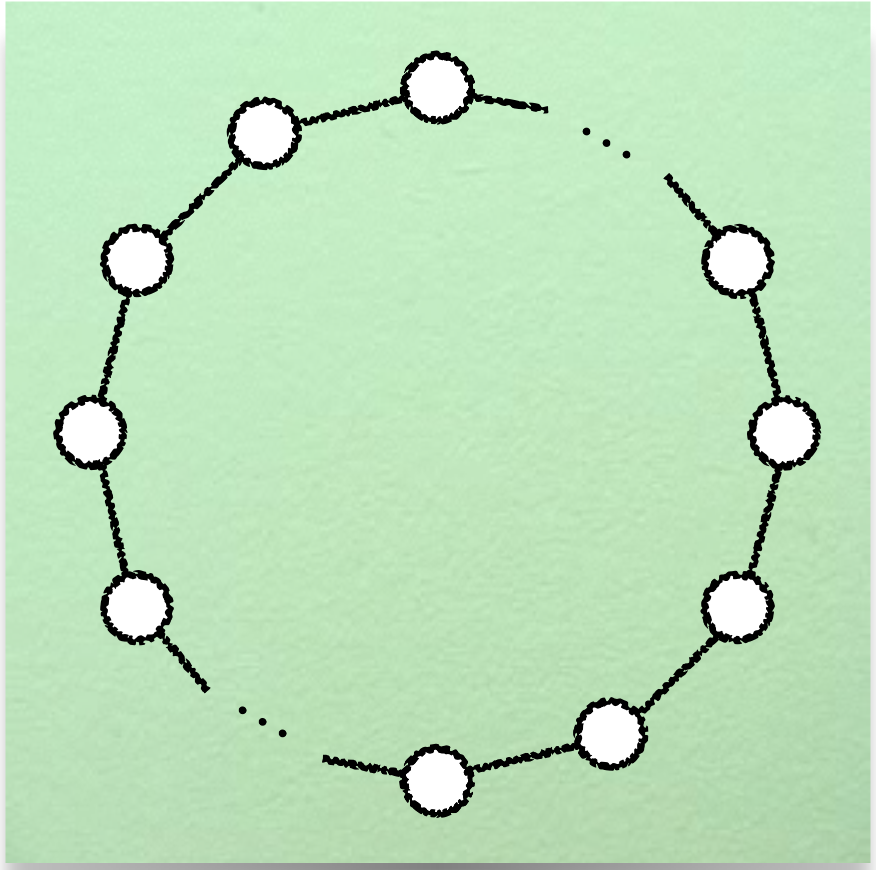

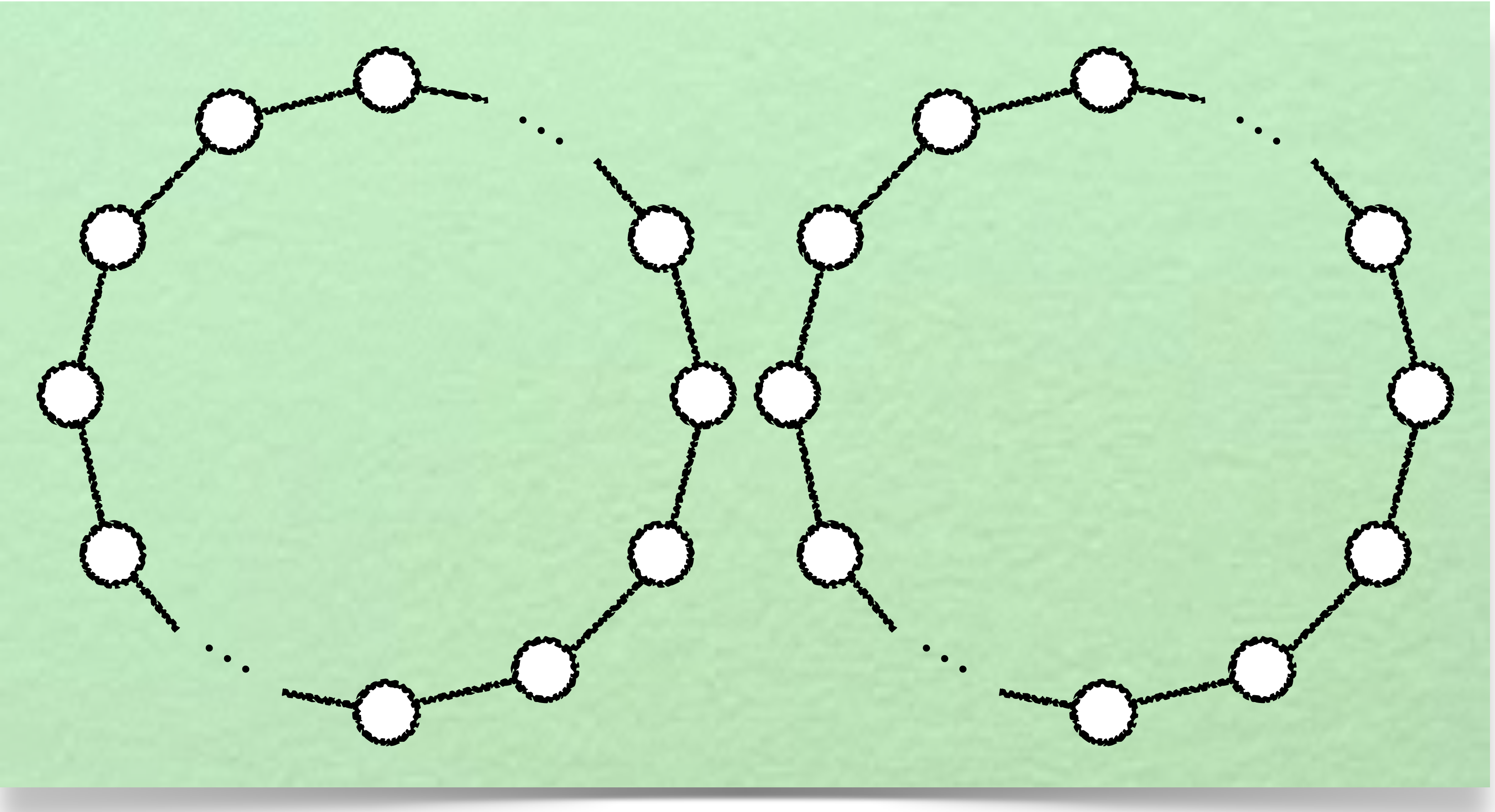

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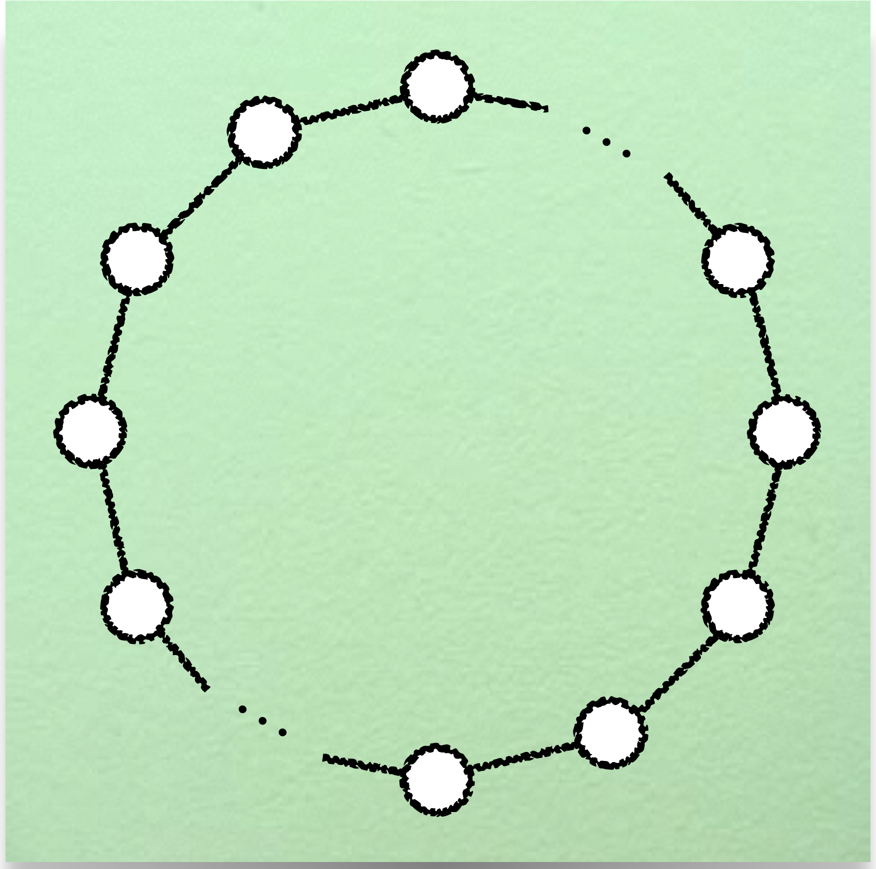

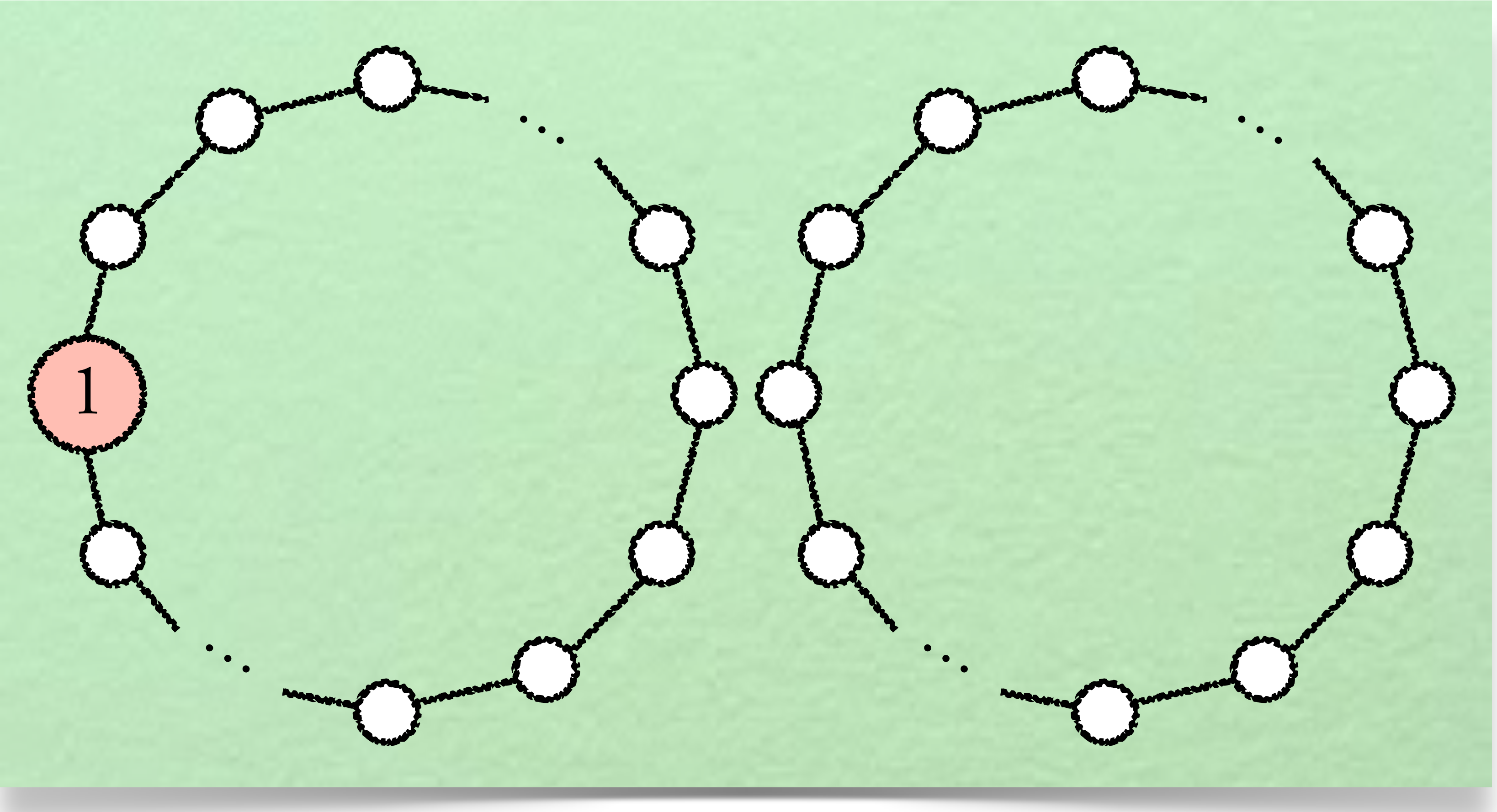

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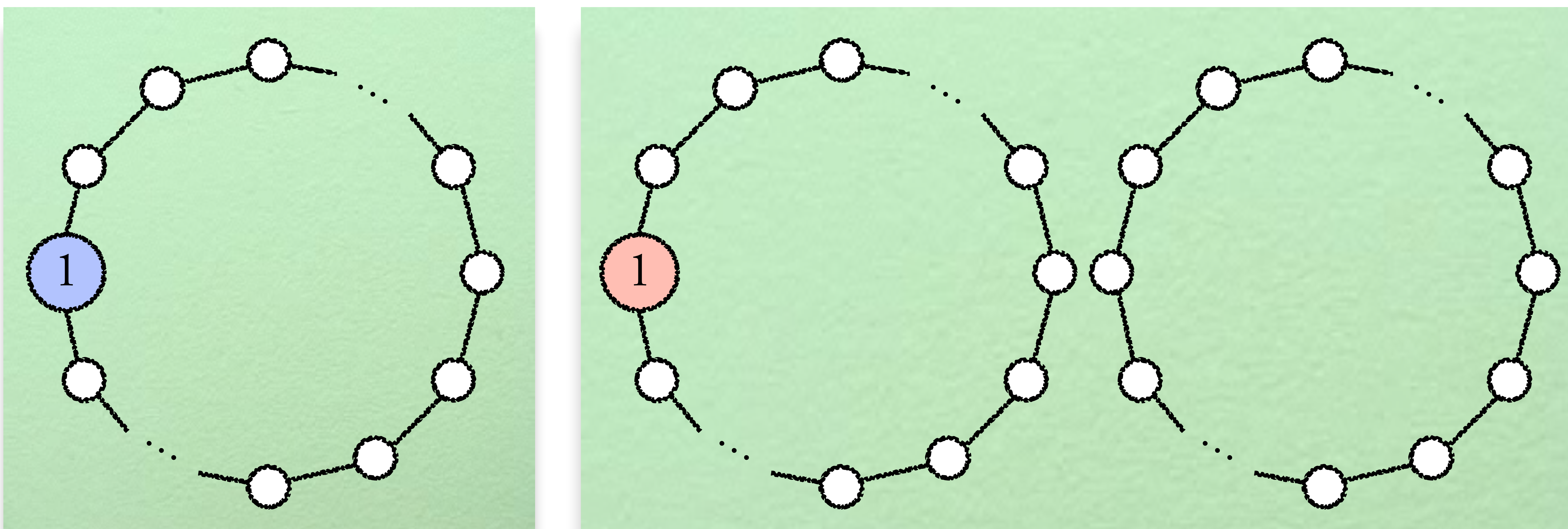

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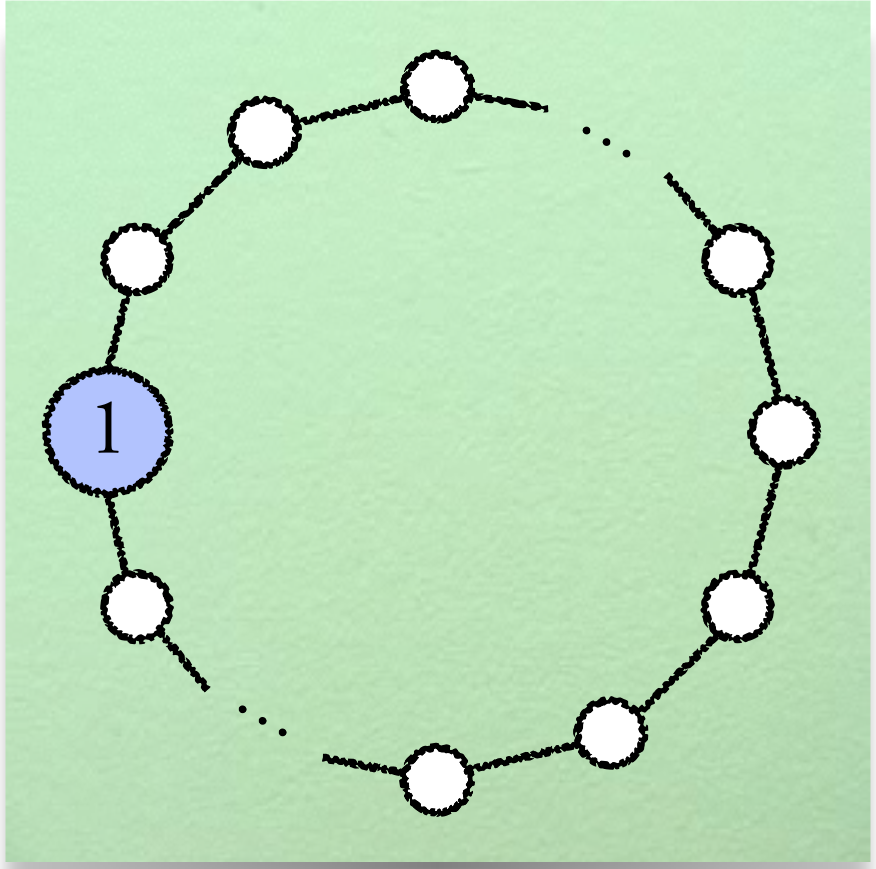

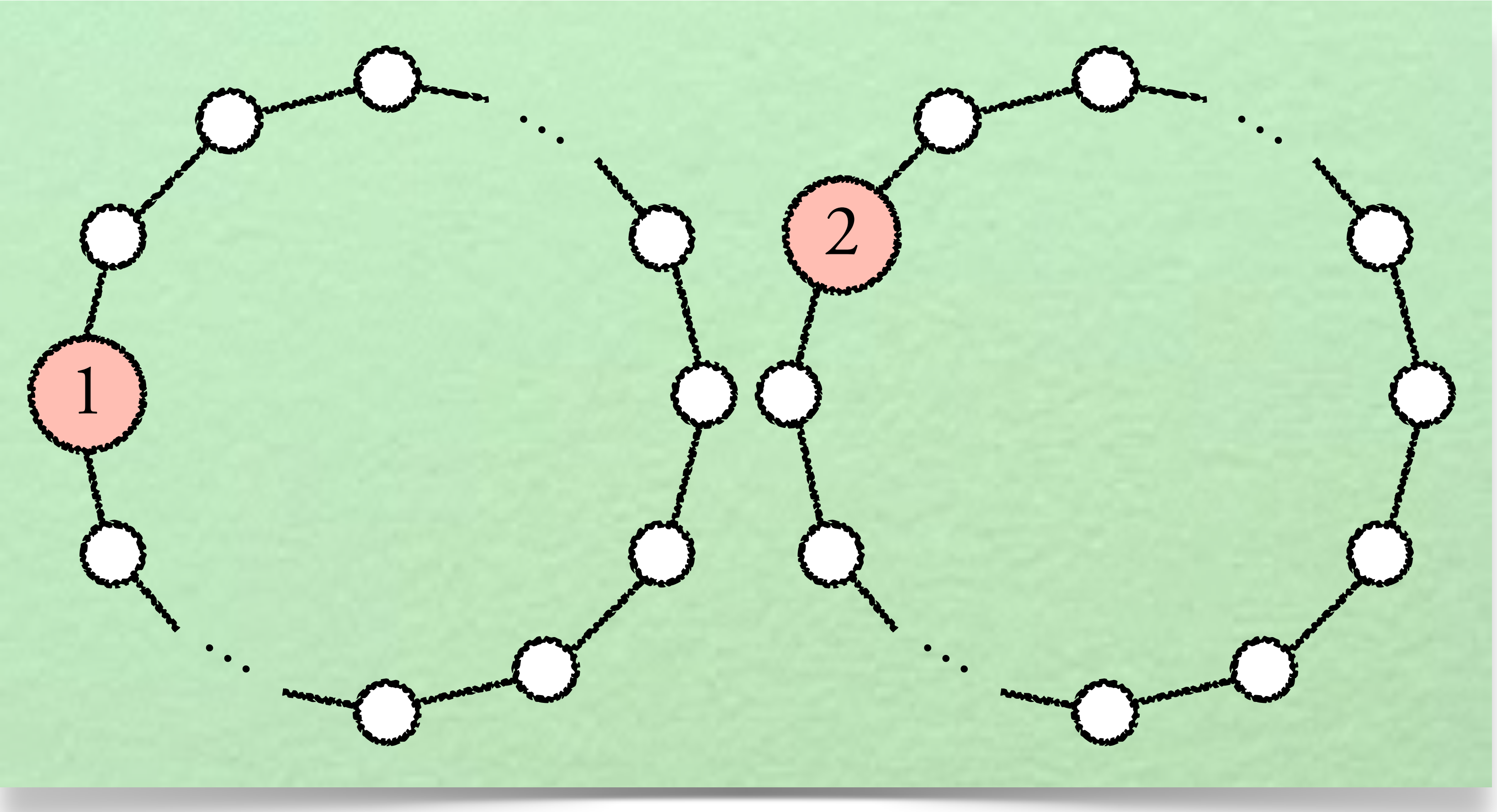

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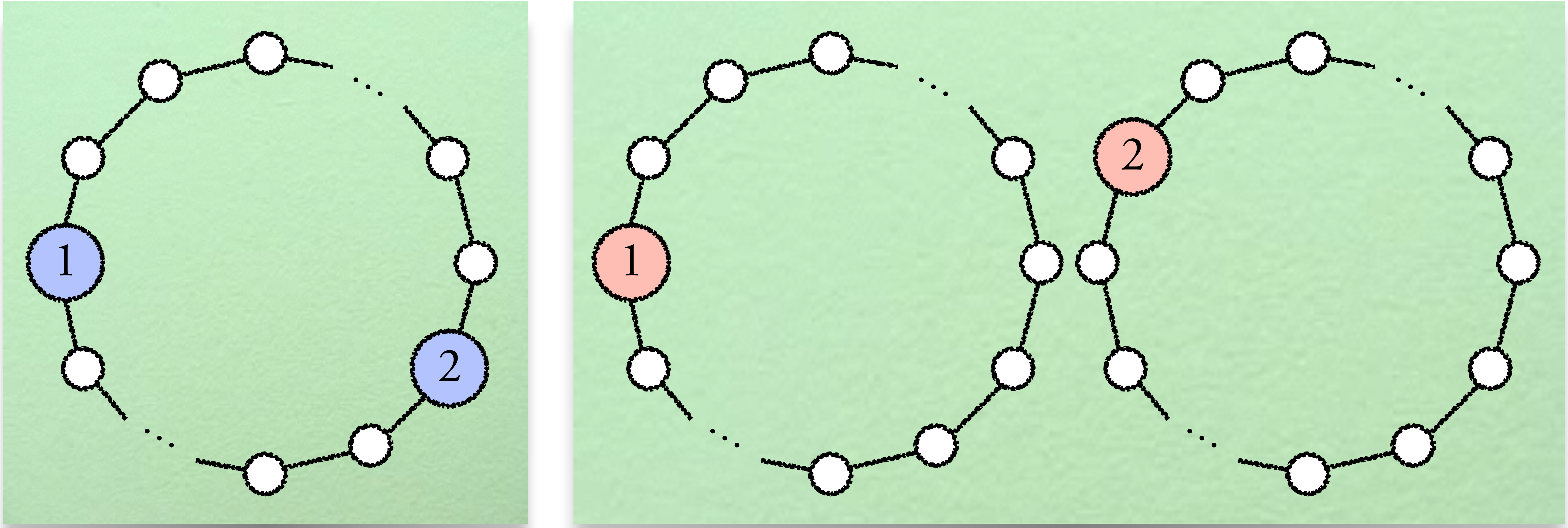

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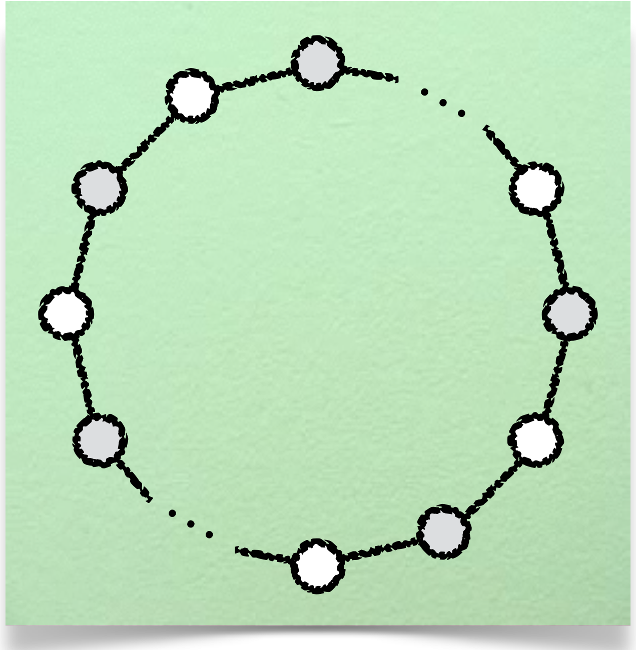

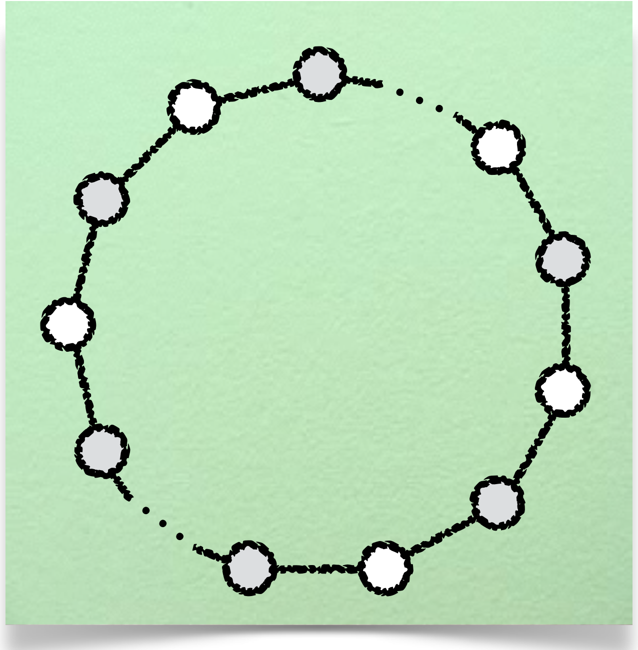

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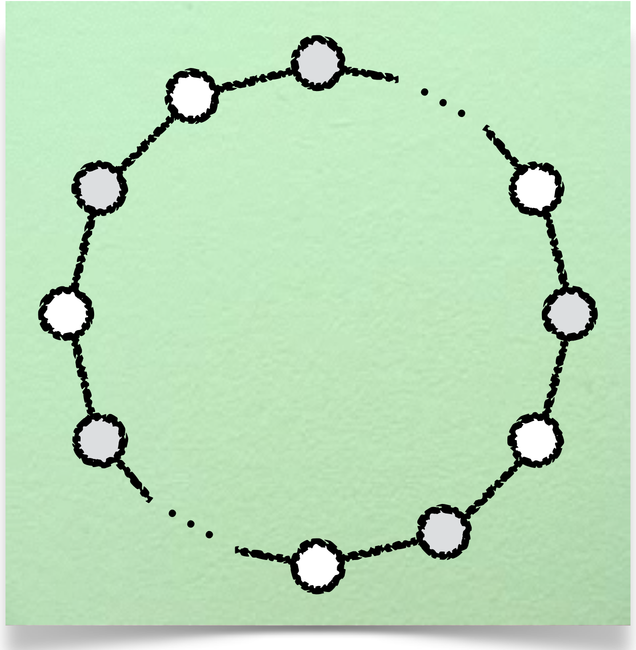

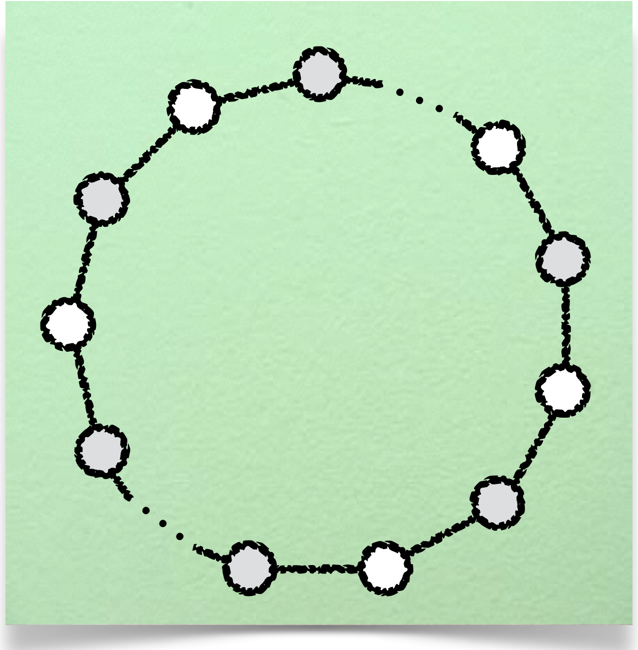

- finiteness
- acyclicity

...

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